

Alphabetic Constructions in Symbolic Dynamics

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In symbolic dynamics we study sequences of symbols. These symbols may represent the state of some system at a certain point in time. A transformation in this space represents the change in the system from one moment to the next. Certain questions about the properties of these transformations are central to the study of any dynamical system. One such property is called minimality, in which the orbit of every point in the space is dense in the space. In this paper, we look at a method for constructing sequences that generate minimal spaces and also include a method whereby any sequence can be interpolated into such a sequence.

Symbolic Dynamics

Let $X = \{0,1\}^{\mathbf{Z}^+}$. That is, if $x \in X$ then x is a function $x(n): \mathbf{Z}^+ \rightarrow \{0,1\}$. In words, X is the set of all one-sided infinite sequences of 0's and 1's. X will be our symbol space.

Definition. Let T be a transformation on X given by $Tx(n) = x(n+1)$. T is called a left-shift. The notation T^jx will represent the usual composition of transformations. The set $\{T^j : j \in \mathbf{Z}^+\}$ is a semi-group of transformations on X .

The pair (X, T) is a dynamical system. It is possible to introduce a topology in which all the transformations T^k are continuous.

Definition. Let $x, y \in X$ and define the distance d by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \max \left\{ \frac{1}{n}, x(n) \neq y(n) \right\} & \text{if } x \neq y. \end{cases}$$

In other words, given $\epsilon > 0$, $d(x, y) < \epsilon$ if and only if there exists N (depending on ϵ) such that $x(n) = y(n)$ for all $n < N$. Clearly $N = \epsilon^{-1}$.

Minimal Sequences

In general, we classify a point $x \in X$ by the type of dynamical system found in the topological closure of the orbit of x under the action of the semi-group of transformations. The term minimal implies that any point in this orbit closure will generate the same dynamical system under the semi-group action. In terms of a single point x , we may offer the following equivalent definition.

Definition. $x \in X$ is minimal if for each $\epsilon > 0$ and for each $j, M \in \mathbf{Z}^+$, there exists $m > M$ such that $d(T^{m+j}x, T^jx) < \epsilon$.

An alternate form of this definition would be: Given j and M , there exists $m > M$ such that $x(n+i) = x(n+i+m)$ for $0 \leq i \leq j$, and any $n \geq 0$. In words, this means that any block of characters found in x will appear infinitely often in x .

Alphabetic Constructions

Definition. Any finite set A of symbols is called an alphabet. A word in A , of length n , is a string of n elements of A .

Because a word is a finite string, it can be thought of as a piece of a sequence. We construct infinite sequences by choosing a succession of words from different alphabets so that each new word is a longer extension of the previous word. This process is easily described using induction.

Initialization:

Let $A_0 = \{0,1\}$ be our initial alphabet and $n_0 = 2$ be chosen as our initial length (in the application, lengths will be chosen in a prescribed manner). Choose an element $x_0 \in A_0$ as the seed of our final sequence.

Note. If the sequence under construction is x , then $x(0) = x_0$.

Inductive step:

Assume a succession of alphabets $A_0 \dots A_k$, lengths $n_0 \dots n_k$, and words $x_0 \dots x_k$ have been defined. Construct A_{k+1} by forming all words of length n_k from A_k whose first character is x_k (additional criteria may be used in this construction). Choose x_{k+1} from A_{k+1} and select $n_{k+1} = 2$ by whatever process is necessary.

Note. x_k is also a word from A_0 of length $N_k = (n_{k-1})(n_{k-2})\dots(n_1)(n_0) = 2^k$. Further, the sequence $x(n)$ is defined for $n \leq N_k$. When x_{k+1} is constructed, $x(n)$ is extended so that it is defined for $n \leq n_k N_k$.

Applications

The ambiguity in the choice of n in the inductive construction of $x(n)$ is clarified by an application of the process. We use this ambiguity in two different sequence constructions.

Application #1

This application constructs a minimal sequence. We begin with a variation of the law of large numbers.

Lemma 1. Let A be an alphabet with m characters. Given $\epsilon > 0$, there exists a length n so that the probability that a random word of length n from A will begin with a specified character and contain all the other characters in A at least once is greater than $\frac{1}{m}(1 - \epsilon)$.

proof The proof follows once we realize that the probability of a word of length n satisfying the given properties is at least

$$\frac{m^n - (m-1)^n}{m^{n+1}} = \frac{1}{m} \left(1 - \left(\frac{m-1}{m} \right)^n \right).$$

We now modify inductive construction of $x(n)$ as follows:

Initialization:

Let $A_0 = \{0,1\}$ be our initial alphabet. Choose an element $x_0 \in A_0$ as the seed of our final sequence. Select $n_0 - 2$, as in Lemma 1, so that every word of length n_0 has x_0 as its first character and contains both 0 and 1 with probability at least $\frac{1}{4}$.

Inductive step:

Assume a succession of alphabets $A_0 \dots A_k$, lengths $n_0 \dots n_k$, and words $x_0 \dots x_k$ have been defined. Construct A_{k+1} by forming all words of length n_k from A_k whose first character is x_k and contain every other character of A_k at least once. Choose x_{k+1} from A_{k+1} . Suppose that A_{k+1} has m_{k+1} characters and select $n_{k+1} - 2$, as in Lemma 1, so that every word of length n_{k+1} from A_{k+1} has x_{k+1} as its first character and contains every other character of A_{k+1} at least once with probability at least $\frac{1}{2m_{k+1}}$.

Theorem. The sequence $x(n)$ constructed in the inductive process is minimal.

proof Using the alternate form of the definition of a minimal sequence, a block $x(n+i)$ for $0 \leq i \leq j$ is part of a character in some alphabet A_k . By the construction, this character, and hence the block will then appear in every word used to form the alphabet A_{k+1} . The consequence of the inductive construction is that the final sequence $x(n)$ can be broken up into letters from A_{k+1} . We need only pick a letter beyond M and the property is satisfied.

Application #2

In this application, we construct a minimal sequence that satisfies the property that values in positions along the sequence are predetermined.

Let $\{e_n\}$ be a sequence of positive integers, and let $y(n) \in X$. If $x(n)$ satisfies the property that $x(e_n) = y(e_n)$ for all n then we say that x agrees with y , along the sequence $\{e_n\}$.

Definition. For any $n \in \mathbb{Z}^+$, define $\kappa(n) := \text{card}\{i : e_i \leq n\}$.

One interesting choice of $\{e_n\}$ is $e_n = n^2$. In this case, $\kappa(n) = \lfloor \sqrt{n-1} \rfloor + 1$. As usual, $\lfloor x \rfloor$ represents the greatest integer function.

Lemma 2. Let A be an alphabet with m characters. If $\lim_{m \rightarrow \infty} \frac{\kappa(m)}{m} = 0$, there exists n so that we may find word of length n from A which contains all characters of A at least once and agrees with $y(n)$.

proof As in the proof of Lemma 1, it is clear that the probability of a word satisfying the first property (notice that the initial character restriction is removed) is at least

$$\left(1 - \left(\frac{m-1}{m}\right)^n\right).$$

On the other hand, the probability that a word satisfies the second property is exactly

$$\frac{m^{n-\kappa(n)}}{m^n} = m^{-\kappa(n)}.$$

If no word satisfies both properties simultaneously then

$$\left(1 - \left(\frac{m-1}{m}\right)^n\right) + m^{-\kappa(n)} \leq 1.$$

It follows that

$$\frac{\kappa(n)}{n} \geq 1 - \frac{\log(m-1)}{\log(m)} > 0$$

for all n . This clearly contradicts the hypothesis so that such a word must surely exist.

Note. Although Lemma 2 establishes the existence of only one word, it follows that other words may be constructed by permuting the letters in the free positions.

Once again we modify the original inductive construction of $x(n)$.

Initialization:

Let $A_0 = \{0,1\}$ be our initial alphabet. Choose an element $x_0 \in A_0$ as the seed of our final sequence. If $e_0 = 0$, we must select $x_0 = y(0)$. Otherwise, the choice is random.

Select

$n_0 - 2$, as in Lemma 2, so that some word of length n_0 contains both 0 and 1 and agrees with y . Rename $y(n)$ as $y_0(n)$ and $\{e_n\}$ as $\{e_{0,n}\}$.

Inductive step:

Assume a succession of alphabets $A_0 \dots A_k$, lengths $n_0 \dots n_k$, sequences $\{e_{k,n}\}$, symbol sequences $y_k(n) \in A_k^{\mathbf{Z}^+}$, and words $x_0 \dots x_k$ have been defined. Construct A_{k+1} by forming all words of length n_k from A_k that contain every character of A_k at least once. Choose x_{k+1} from A_{k+1} as in Lemma 2 so that x_{k+1} agrees with $y_k(n)$. Define $\{e_{k+1,n}\}$ from $\{e_{k,n}\}$ by

$$e_{k+1,n} = \left\lfloor \frac{e_{k,n}}{n_k} \right\rfloor.$$

The relationship between $\{e_{k+1,n}\}$ and $\{e_{k,n}\}$ is linear so that $\lim_{m \rightarrow \infty} \frac{\kappa(m)}{m} = 0$ is preserved.

(κ is a function specific to the sequence $\{e_{k,n}\}$ in question. It serves to mark the number of free positions we may use in the construction. Because it is only used in Lemma 2, subscript notation has not been adopted.) $y_{k+1}(n)$ is defined from $y_k(n)$ by partitioning $y_k(n)$ into blocks of length n_k and replacing them with appropriate characters from A_{k+1} . Suppose that A_{k+1} has m_{k+1} characters and select $n_{k+1} - 2$, as in Lemma 2, so that some word of length n_{k+1} from A_{k+1} contains every character of A_{k+1} at least once and agrees with $y_{k+1}(n)$.

This final construction is important because one could ask the question: "If an observer does not record the state of the system at every moment, can the total system be determined?" If the moments occur at regular intervals the answer may be yes. In general, the answer is no. As a result of our construction we may offer a counter-example. Form $y(n)$ so that in positions corresponding to perfect squares we fix values from a sequence that is not minimal, then fill in the rest of the values randomly. Following the construction in Application #2, we may construct $x(n)$ so that $x(n^2) = y(n^2)$ and $x(n)$ is minimal. In such a system, an observer looking at the moments corresponding to perfect squares would get a completely incorrect impression.