INVESTIGATING HISTORICAL PROBLEMS USING *GEOMETER'S SKETCHPAD*

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Naturally, history has a place in the mathematics classroom that should not be overlooked. What many mathematicians fail to recognize is the enhancement of historical investigations by use of technology. *Geometer's Sketchpad*, a dynamic and interactive piece of software, provides a work environment that allows one to create, test, validate, and manipulate objects. It has the power and flexibility to allow students to examine an infinite number of situations, instead of one singular static case, which is invaluable in attempts to make mathematical conjectures and generalizations. The purpose of this paper is not to shed new light on tasks or problems related to history of math, but to share "golden" opportunities where use of *Geometer's Sketchpad* (GSP) enhances the investigation of many famous geometric problems. The scope of situations to investigate with this software are unlimited. Users quickly see how technology often generates many additional questions or tasks for students to explore, as well as enabling them to visualize the connections among various mathematics topics.

Students could easily begin a series of geometric problems with one of the famous problems of antiquity: the Pythagorean theorem, the golden ratio, or Archimedes' Arbelos. These three problems are ideal situations to investigate using GSP. The history of the Pythagorean theorem, probably one of the most well-known theorems in mathematics, illustrates that there was evidence of the ancient Chinese investigating this theorem even before the Babylonians. To this date, it is believed that there are approximately 370 proofs of this theorem - even one from past United States President, James Garfield.

GSP's construction tools and capabilities allow one to investigate the Pythagorean theorem in any number of ways. One example could be to construct any right triangle and generate squares on the legs and hypotenuse. This, in turn, makes it easy to manipulate the right triangle and compare the areas of the generated squares. The power of technology when investigating such a theorem is that by using GSP one can test an infinite number of cases at any given moment. Figure 1(a) & Figure 1(b) provide the reader with a small sample of possible results that could be generated.

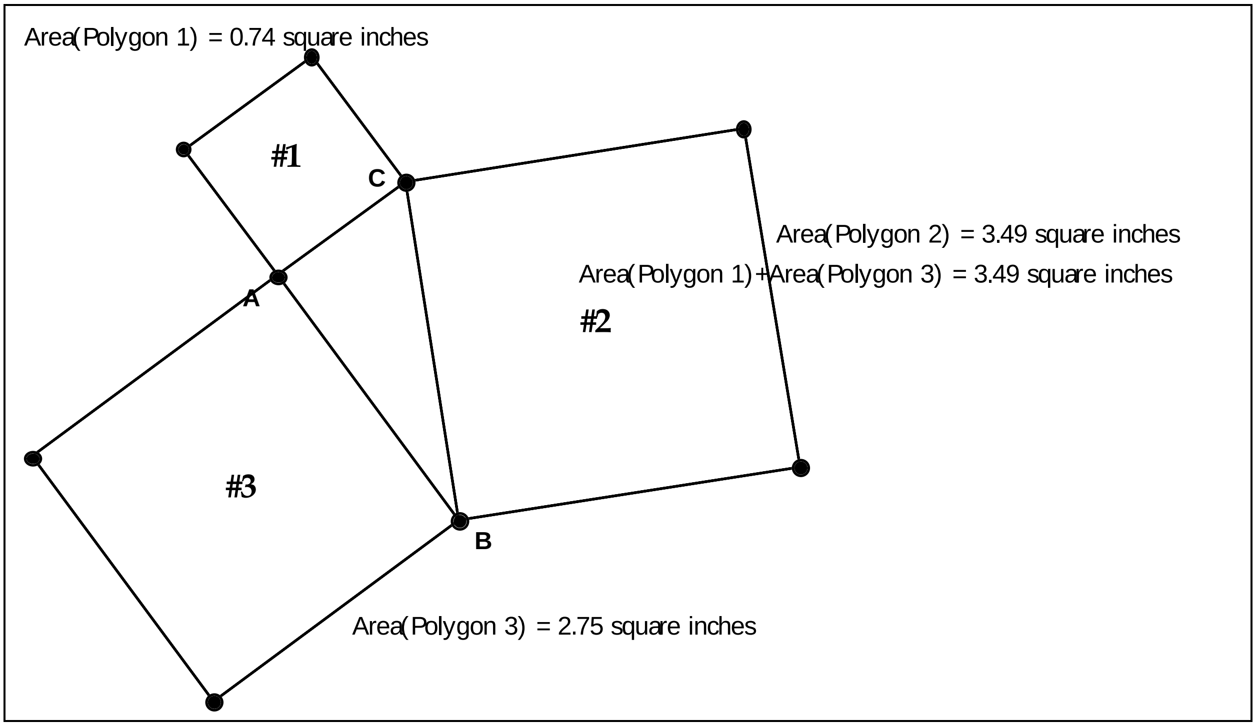

Figure 1(a): Pythagorean Theorem

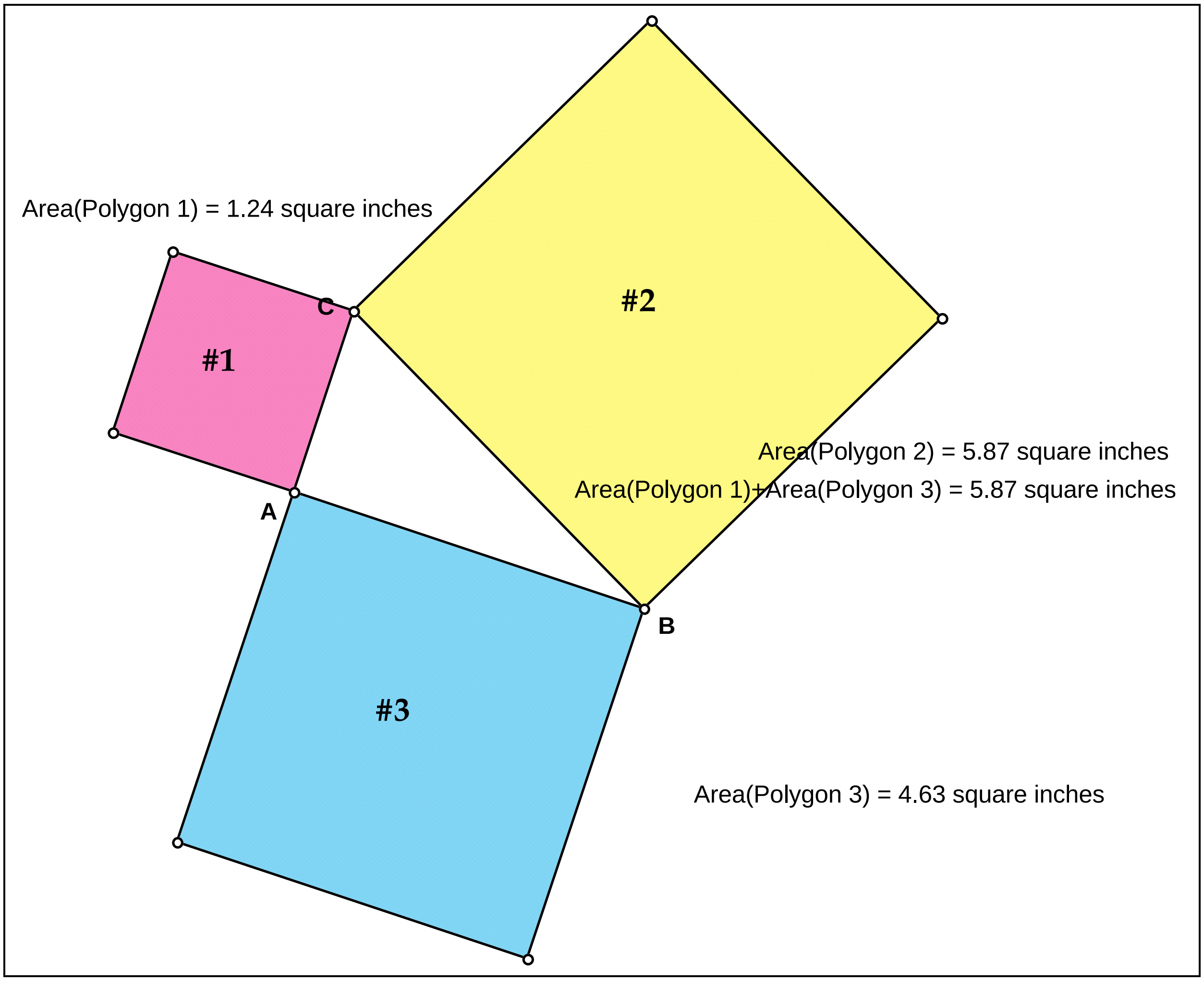

Figure 1(b): Pythagorean Theorem

Once students have explored the situation using squares generated on the triangle legs and hypotenuse, new situations and problems to investigate may emerge:

- What if you generate equilateral triangles on the legs and hypotenuse?
- What if you generate rectangles? semi-circles?
- Can these situations (or others) be used to help prove the theorem? How?

GSP also provides a dynamic environment for students to explore any number of situations to test other triangles that are not right triangles. Students quickly see how the technology can be used to visually gather information in order to make attempts to prove the theorem.

Other problems that lend themselves nicely to explorations involving the use GSP are geometry theorems involving triangles or quadrilaterals. The first, Ceva's Theorem, deals with the centroid of a given triangle or center of gravity. It states:

Three lines drawn from vertices A, B, and C of triangle ABC meeting the opposite sides in points L, M, and N respectively, are concurrent if and only if $\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1$.

One might note that one case for L, M, and N is when they are the midpoints of the opposite sides (Figure 2(a)). Obviously, this would result in equal products, which indicates the lines are concurrent.

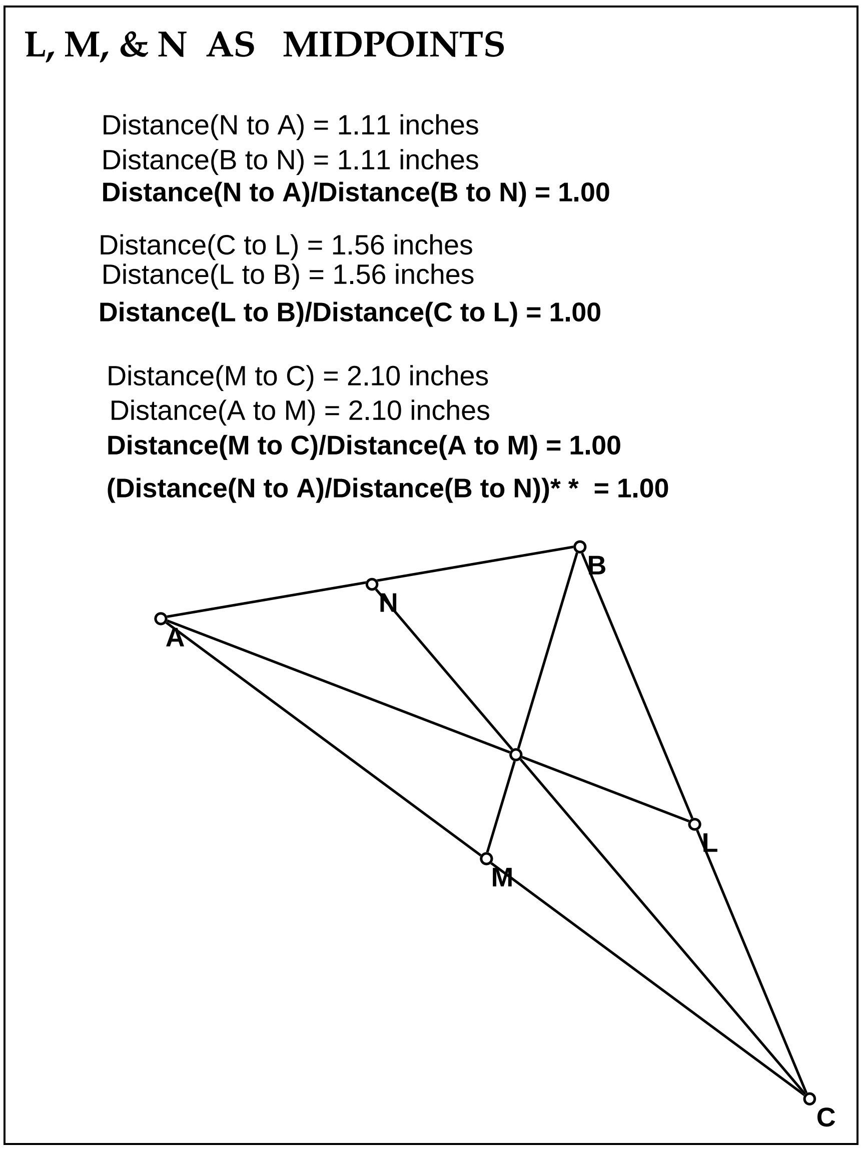

Figure 2(a): Ceva's Theorem using midpoints

GSP provides an interactive environment for students to animate points L, M, and N on the respective sides and calculate distinct products of ratios. The technology allows students to visually see what occurs as the ratios change with respect to various positions of L, M, and N and different triangles. Figures 2(b) and 2(c) are examples to help illustrate this animation process for Ceva's theorem.

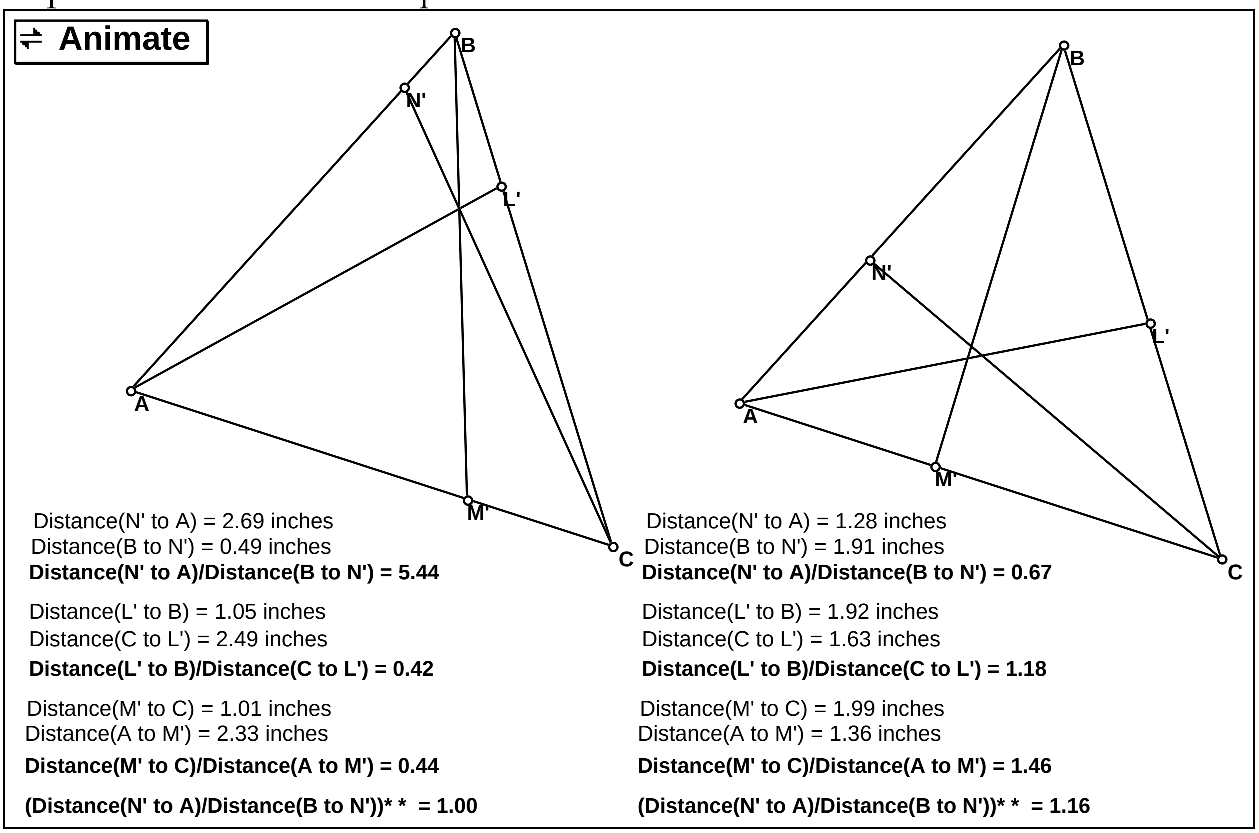

Figure 2(b): L, M, N not as midpoints

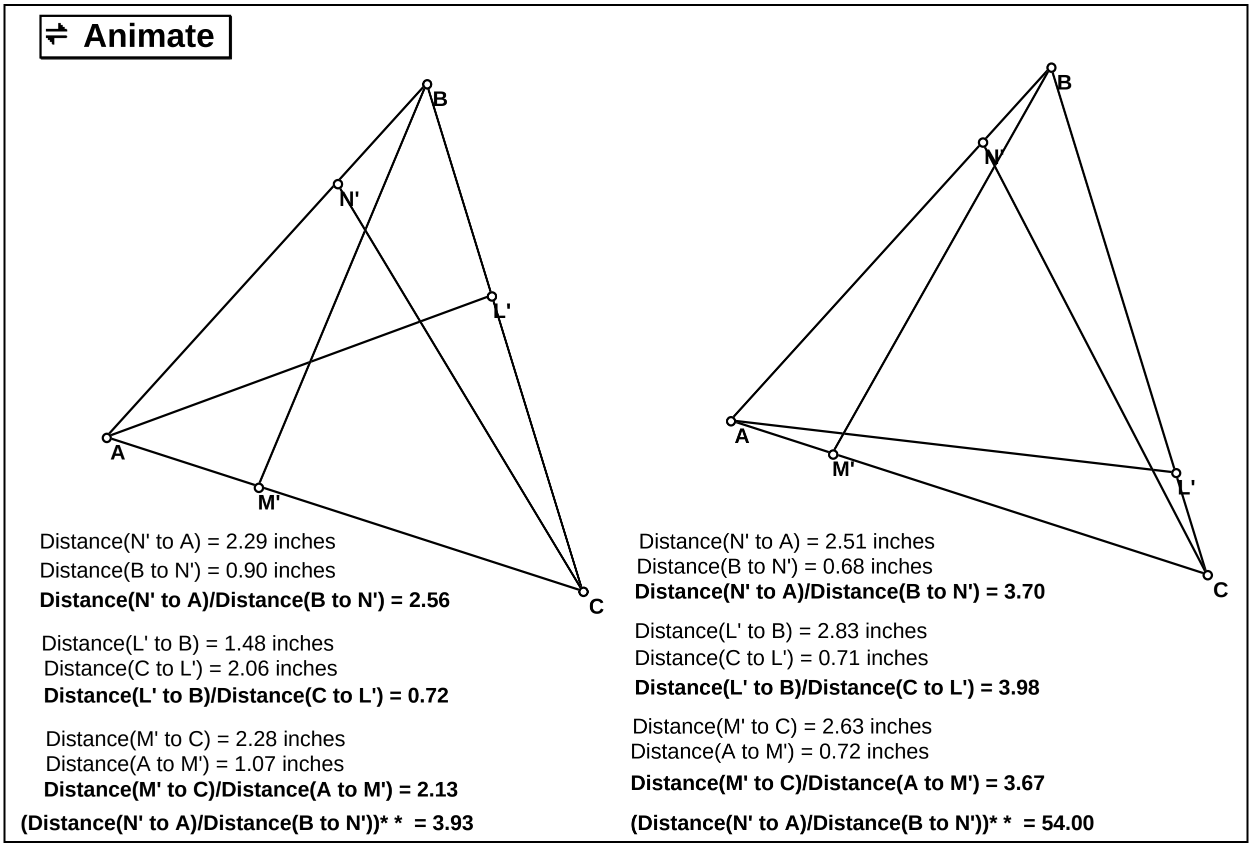

Figure 2(c): Ratio products > 1

Due to the features the technology offers, a number of additional questions and problems for students to explore, surface.

- What do you notice about the given situation if you generate the three angle bisectors of the triangle?
- How do the angle bisectors relate to the triangle's inscribed circle?
- Does it matter what kind of triangle you start with? Do different triangles give you different information? Why or why not?
- How can a different theorem be used to gather information to help prove Ceva's theorem?

Ptolemy's Theorem, related to quadrilaterals, is another excellent theorem that lends itself nicely to explorations using GSP. The theorem states:

In a cyclic (inscribed) quadrilateral, the product of the lengths of the diagonals is equal to the sum of the product of the lengths of the pairs of opposite sides.

This theorem is quite useful in trying to find the measure of a diagonal of a quadrilateral if we know it is cyclic. GSP is an excellent environment for one to test whether a quadrilateral is cyclic or not. Once this characteristic has been identified, Ptolemy's theorem can be used to find the length of the diagonal of the given quadrilateral. We can also use Ptolemy's theorem to determine a quadrilateral's cyclic character. Seeking out quadrilaterals that are cyclic provides a nice exploration of this theorem and can easily be examined using GSP (Figure 3(a)). Students are provided with a situation where they can make generalizations about various quadrilaterals and their unique properties and look for patterns among cyclic quadrilaterals.

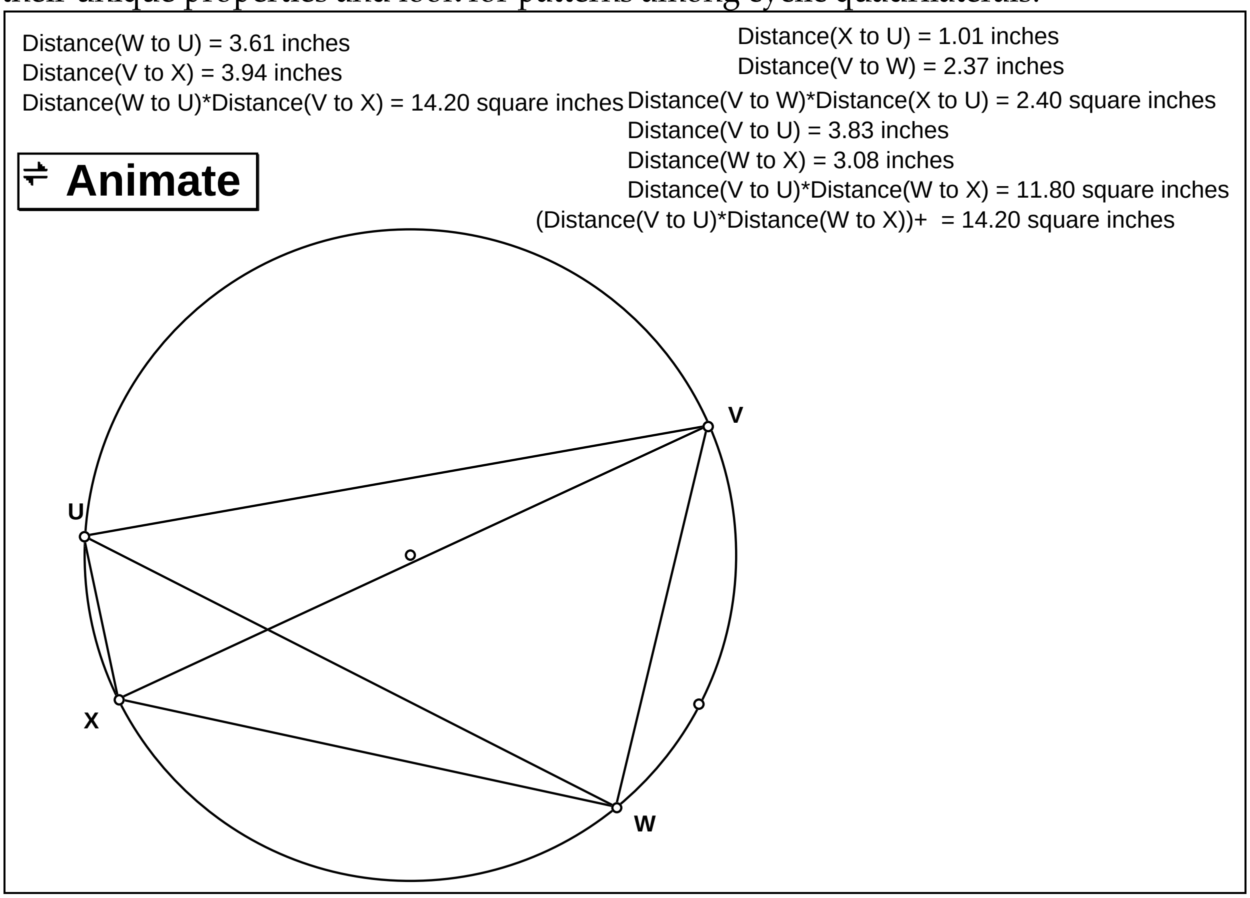

Figure 3(a): Ptolemy's theorem applied to any quadrilateral

Ptolemy's theorem has a rather elegant connection with another famous geometric theorem. When Ptolemy's theorem is applied to any given rectangle, the situation collapses into the Pythagorean theorem (Figure 3(b) & 3(c)). The visual geometric representation supports the numerical data that is generated and results in the relationship of $a^2 + b^2 = c^2$.

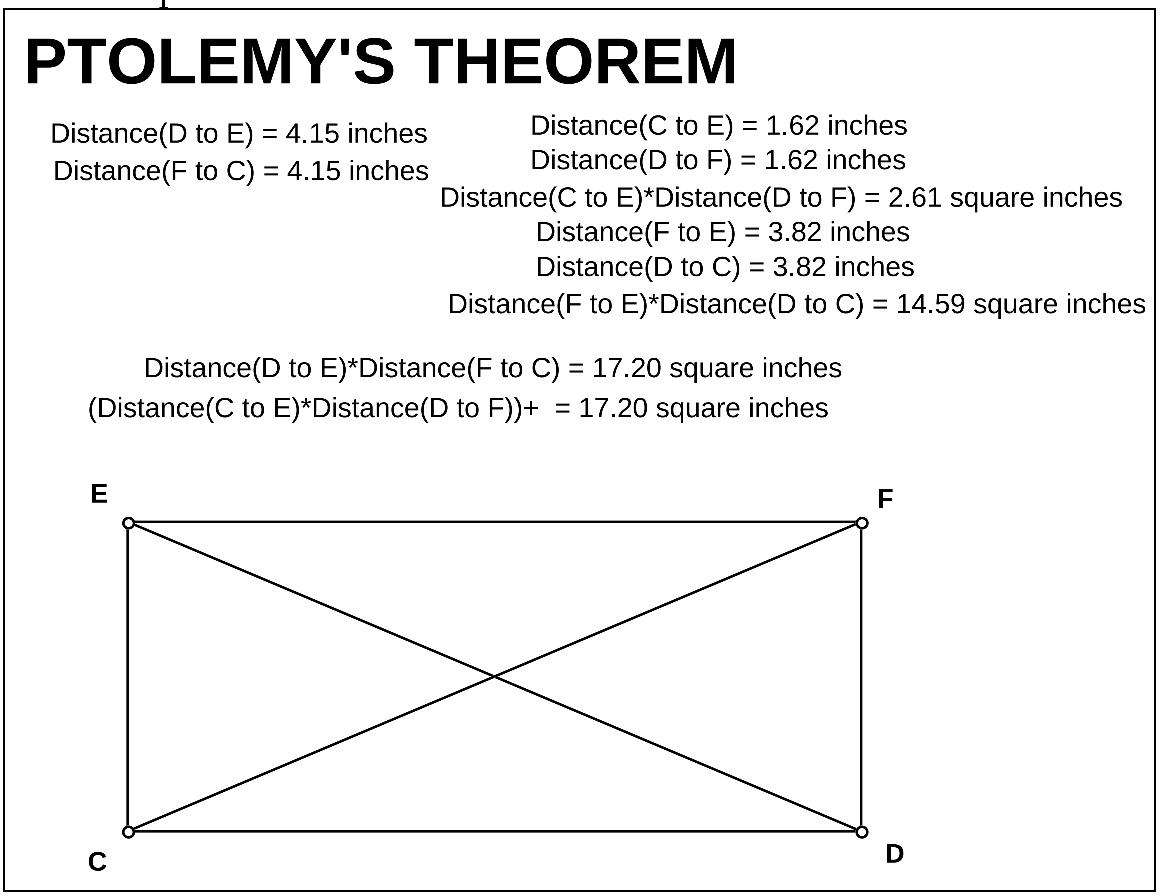

Figure 3(b): Rectangle situation #1

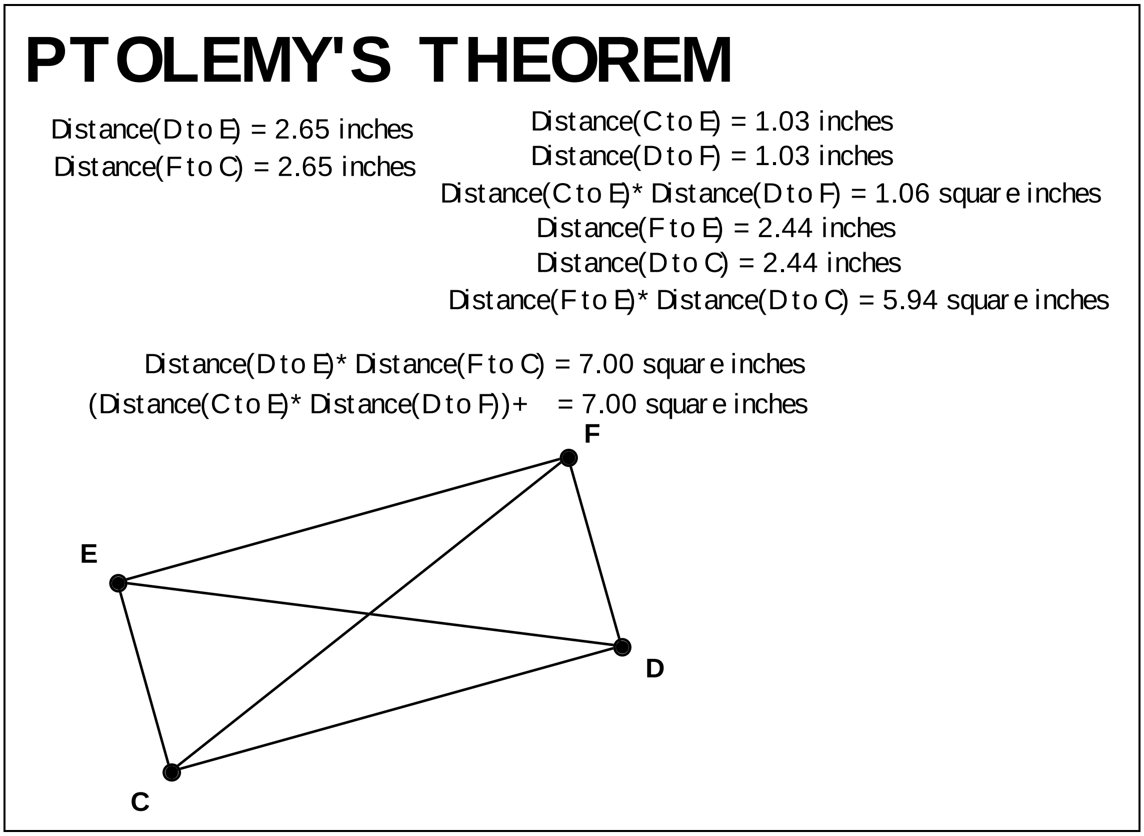

Figure 3(c): Rectangle situation #2

Simson's Theorem, actually discovered by a man named Wallace (1797), states:

The feet of the perpendiculars drawn from any point on the circumcircle of a given triangle to the sides of the triangle are collinear.

The following figures illustrate the collinear points, X, Y, and Z, formed with different triangles ABC. The point on the circumcircle of the given triangle is identified as point P.

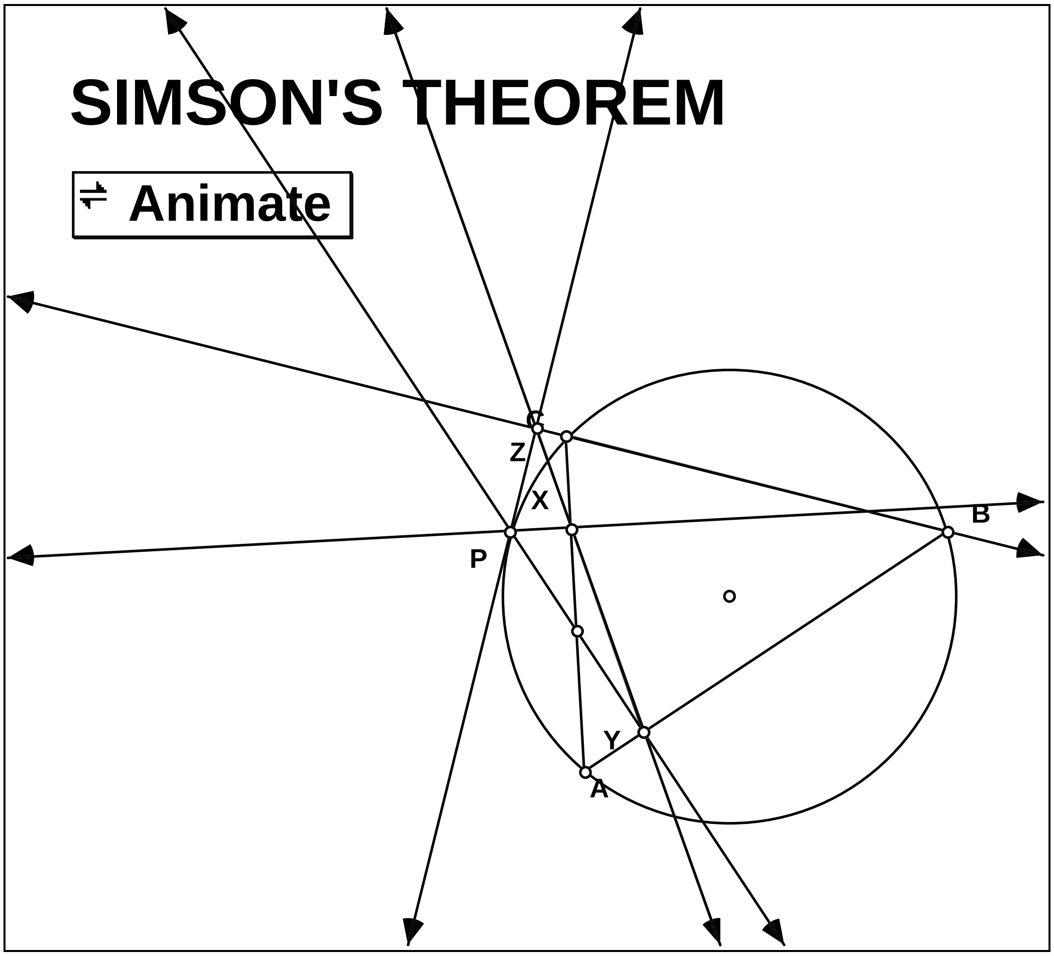

Figure 4(a): Given any arbitrary triangle

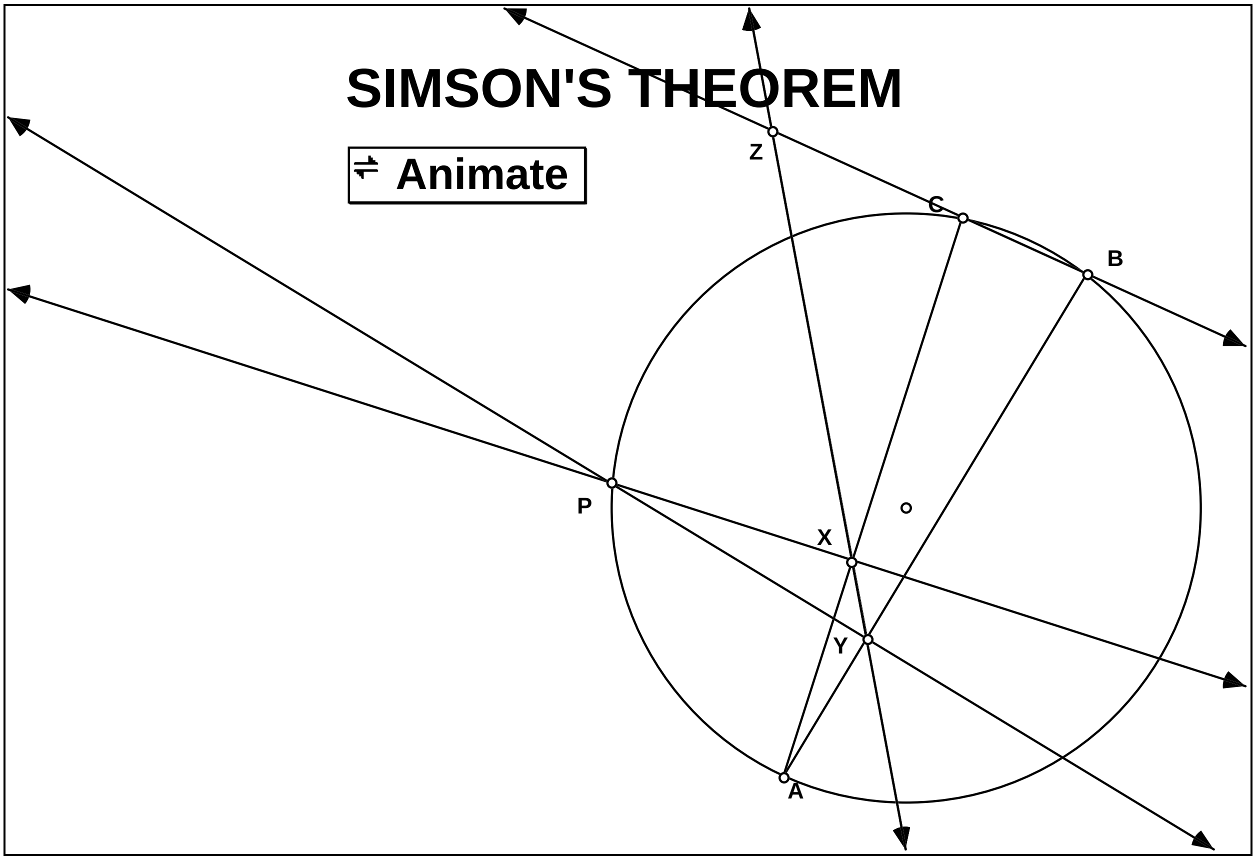

Figure 4(b): Given isosceles triangle ABC

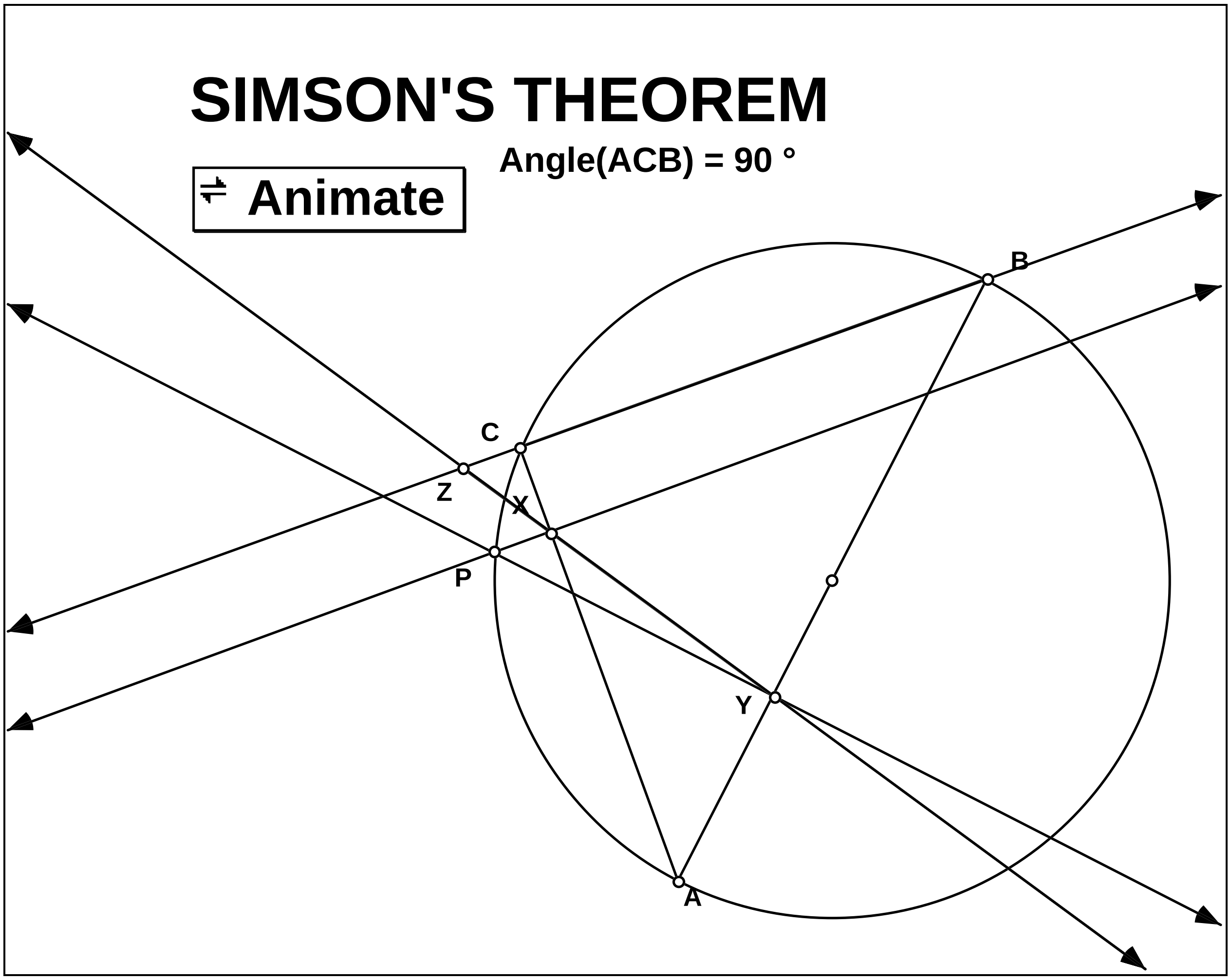

Figure 4(c): Given right triangle ABC

Other questions that could surface as a result of investigating Simson's theorem are:

- How many Simson lines can a triangle possess?
- Will the Simson line be the same for all triangles that are of the same type (i.e.: all equilateral triangles? all right triangles?)? Why or why not?
- What if the original point is chosen to be one of the triangle's vertices? How does this effect the Simson line?

It is no surprise that many notable world leaders had a keen interest in mathematics. Napoleon Bonaparte, French leader and emperor, was supposed to have discovered a geometric construction. A statement of the construction is as follows:

Draw any scalene triangle. On each side of the triangle construct an equilateral triangle facing outwards. Then find the center of the circumscribed circle of each equilateral triangle. Now connect the centers to form another triangle. This last triangle is also equilateral.

An example of the construction follows. The shaded triangle is the given scalene triangle. The identified measurements are for the segments of the generated equilateral triangles and finally, the equilateral triangle formed by the centers of the circles.

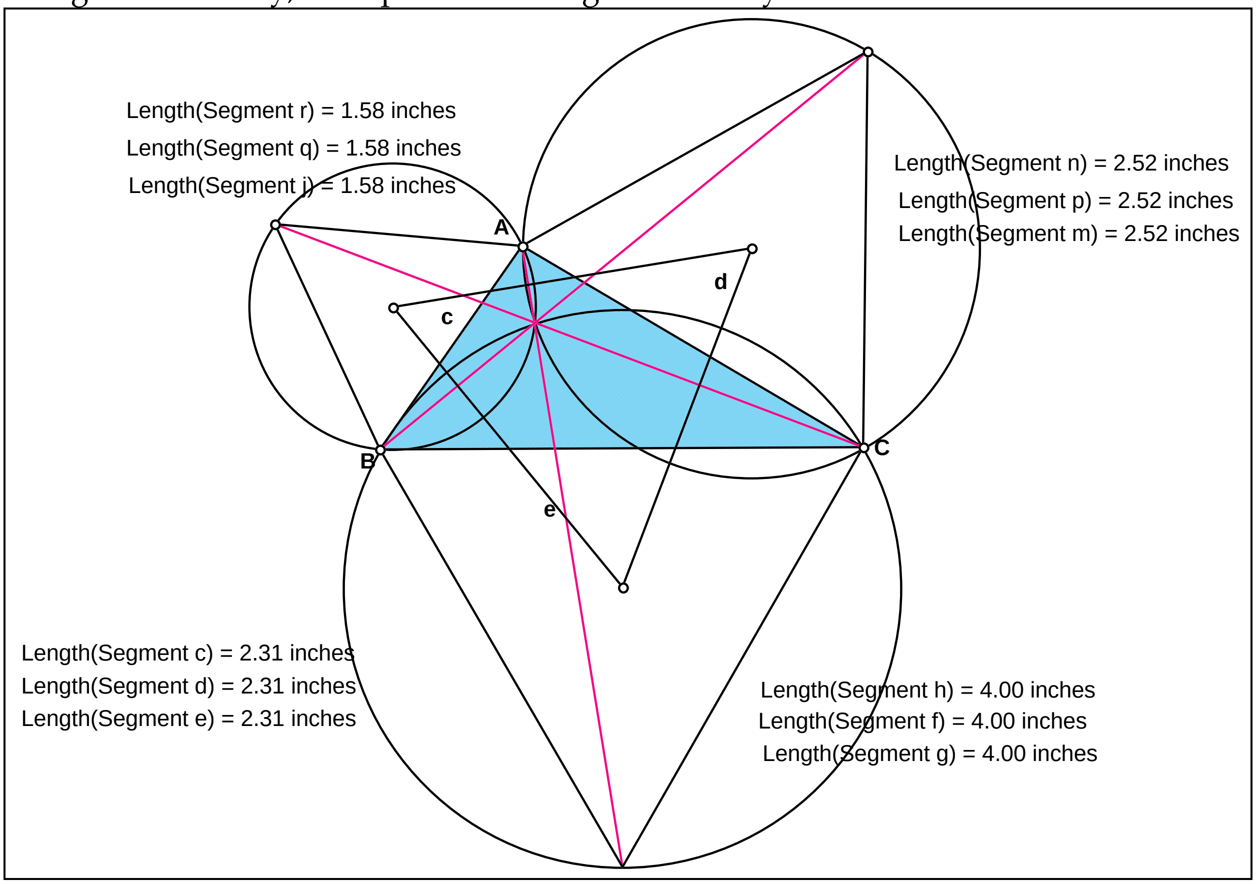

Figure 5: Napoleon's theorem

Other questions or extensions of this construction could be:

- What happens if one begins with a triangle that is not scalene - say isosceles or equilateral? Is the last triangle still equilateral?
- What if one begins with a quadrilateral? What is the final/last polygon?

- Segments formed from the vertices of the original scalene triangle and the vertex of the equilateral triangles that were not part of the scalene triangle are concurrent. In order to make an attempt at proving this, one may need to draw circumscribed circles to the equilateral triangles. How can the centers of these circles be found?
- The point of concurrency is called the equiangular point of the original triangle. Does this change for various triangles? Why or why not?

The possible number of historical problems to investigate using technology is unlimited. Various interactive software and calculators allow teachers and students to pose problems that were once somewhat static in their presentation. As a consequence of using technology to solve historical problems, solving or working with the initial problem often becomes secondary due to the much wider range of questions that could potentially emerge. As a result of these extensions, the mathematical activity has the potential to be on-going. Furthermore, connections among numerical, algebraic, and geometric models become even more prevalent and valuable in investigating historical problems with technology.

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