CYCLES OF A FAMILY OF DIGIT FUNCTIONS

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The main results of this paper (Propositions 1 and 2) will appear in The Journal of Recreational Mathematics, so the proof of Proposition 1 will be omitted here.

Let n be a positive integer with base-10 representation:

$$n = d_{m-1}10^{m-1} + d_{m-2}10^{m-2} + ... + d_1 10 + d_o$$

where the d_i's are the digits of n.

For each positive integer k, define a function g_k on n as the sum of the d_i powers of k:

$$g_k(n) = k^{dm-1} + k^{dm-2} + ... + k^{d1} + k^{do}.$$

The g_k's are the family of digit functions referred to in the title. Other families of digit functions have been studied and will be mentioned briefly later. The primary question of our investigation will be: What happens when we iterate g_k on n?

For example let k = 2 and n = 467. Then

$g_2(467) = 2^4 + 2^6 + 2^7 = 208$

$g_2(208) = 2^2 + 2^0 + 2^8 = 261$

$g_2(261) = 2^2 + 2^6 + 2^1 = 70$

$g_2(70) = 2^7 + 2^0 = 129$

$g_2(129) = 2^1 + 2^2 + 2^9 = 518$

$g_2(518) = 2^5 + 2^1 + 2^8 = 290$

$g_2(290) = 2^2 + 2^9 + 2^0 = 517$

$g_2(517) = 2^5 + 2^1 + 2^7 = 162$

$g_2(162) = 2^1 + 2^6 + 2^2 = 70.$

So, after 3 iterations of g_2, the integer n = 467 enters the 6-cycle (70,129,518,290,517,162).

If k = 3 and n = 278, then

$g_3(278) = 3^2 + 3^7 + 3^8 = 8757$

$g_3(8757) = 3^8 + 3^7 + 3^5 + 3^7 = 11178$

$g_3(11178) = 3^1 + 3^1 + 3^1 + 3^7 + 3^8 = 8757$.

So, after 1 iteration of g_3, the integer n = 278 enters the 2-cycle (8757,11178).

Of particular interest are the 1-cycles (or fixed points) of the g_k's. For example, with k = 3 and n = 221,

$g_3(221) = 3^2 + 3^2 + 3^1 = 21$

$g_3(21) = 3^2 + 3^1 = 12$

$g_3(12) = 3^1 + 3^2 = 12$.

So, after 2 iterations of g_3, the integer n = 221 reaches the 1-cycle (12).

It turns out that for any integer n, with a sufficient number of digits, $g_k(n) < n$. This implies that, under repeated iteration of g_k, every integer n must eventually enter a cycle.

PROPOSITION 1. Let m be the number of base-10 digits of n. If m > B, then $g_k(n) < n$, where

$$B = [(1 + 9\log k)/(1 - c)] \qquad (*)$$

with the brackets representing the greatest integer function and $c = (\log e)/e$.

We are interested in the eventual cycle destination of the sequence $g_k(n)$, $g_k^2(n)$, $g_k^3(n)$, ... Proposition 1 shows that eventually this sequence will reach a point where $g_k^i(n)$ will have no more than B digits, where B is defined in (*). Consequently, we will only need to know the effect of repeated iteration of g_k on integers with no more than B digits.

COROLLARY 1. To find the cycles of the digit functions g_k, it is only necessary to find the cycle destinations of integers n with no more than B digits, where B is defined at (*). In particular, we have:

k	1	2	3	4	5	6	7	8	9	10
B	1	4	6	7	8	9	10	10	11	11

THE COMPUTER PROGRAM

Since the order of the digits of n does not affect the value of the digit function g_k, one only needs to consider unordered sets of digits with repetitions allowed. In an 11 digit search, this reduces the number of cases from 100,000,000,000 to 352,715 -- a critical savings.

The function is iterated on a starting set of digits, extracting a new set of digits at each step and checking to see if one of the known cycle destinations has been reached. After 200 iterations, if no known cycle destination is reached, the current value is printed out for further investigation. If necessary, new cycle destinations are added to the program.

PROPOSITION 2. If 1 <= k <= 10, the ultimate destination of any base-10 integer n under repeated iteration of the function g_k is one of the cycles represented below. The complete cycle can be obtained by iterating g_k on the given cycle representative (rep).

k	Cycle Length	Cycle Rep
1	1	1
2	2	148
2	5	98
2	6	70
2	15	5
3	1	12
3	2	8,757
3	13	54
3	13	118
3	14	32
4	1	4,624
4	1	595,968
4	4	5,268
4	15	36,929
5	1	3,909,511
5	2	1,959,655
5	3	81,886
5	3	1,954,031
5	6	17,031
5	20	81,902
6	2	49,473
6	2	1,967,408
6	14	55,800
6	50	9,338
7	1	13,177,388
7	4	6,002,858
7	44	19,260
8	1	1,033
8	2	41,498
8	3	19,402,896
8	4	299,593
8	9	41,561
8	18	66,250
8	22	4,747
8	88	5,696
9	1	10
9	4	96,788,601
9	10	6,669
9	13	43,119,713
9	23	185,897
9	24	597,080
9	61	605,161
9	91	60,185
10	2	21
10	6	22

FREQUENCY OF CYCLE OCCURRENCE

During the execution of the computer program, one can keep track of the "frequency of occurrence" of each cycle destination in the following sense. The program is executed on unordered sets of up to B digits (with repeated digits permitted), so it can keep track of how many of these sets reach each cycle destination under repeated iteration of g_k.

Some examples:

k	B	Cycle Length	Cycle Rep	Frequency	
4	7	1	4,624	12	
		1	595,968	2	
		4	5,268	7,610	
		15	36,929	11,824	19,448
5	8	1	3,909,511	11	
		2	1,959,655	704	
		3	81,886	118	
		3	1,954,031	23,898	
		6	17,031	366	
		20	81,902	18,661	43,758
8	10	1	1,033	1	
		2	41,498	29	
		3	19,402,896	54	
		4	299,593	1,612	
		9	41,561	186	
		18	66,250	2,304	
		22	4,747	1,002	
		88	5,696	179,568	184,756

OTHER DIGIT FUNCTIONS WHOSE CYCLES HAVE BEEN INVESTIGATED

$p(n)$ = the product of the digits of n

e.g. $p(432) = 4 \times 3 \times 2 = 24$

$f_k(n)$ = the sum of the kth powers of the digits of n

e.g. $f_3(432) = 4^3 + 3^3 + 2^3 = 99$

$f(n)$ = the sum of the factorials of the digits of n

e.g. $f(432) = 4! + 3! + 2! = 32$

g(n) = the sum of the digit-to-digit powers of n

e.g. $g(432) = 4^4 + 3^3 + 2^2 = 287$

Note: These investigations can be (and some have been) done in other bases using base-b digits of the integer in the various functions and extracting the base-b digits of the result.

References:

These papers contain other related references:

D.C. Morrow, Variations on Perfect Digital Invariants, J. Recreational Mathematics, 27:1, pp. 9-12, 1995.

D.C. Morrow, Recurring Digit-to-Digit Invariants, J. Recreational Mathematics, 27:2, pp. 154-156, 1995.