

Boundary Rooting In The Stochastic Traveling Salesperson Problem

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A modified version of the stochastic traveling salesperson problem is described and used to prove a rate of convergence result.

1. Introduction. Consider a special finite collection \mathcal{C} of graphs. (Here vertices will be points in Euclidean space, and an edge between two vertices will be the line segment between the two points.) Associated with each graph in \mathcal{C} is a number which is the sum of its edge lengths. A common optimization problem is to find the minimum such sum. Well-known examples include the traveling salesperson problem, which asks for the length of a minimal cycle through a collection of vertices, and the minimal spanning tree problem, which is self-explanatory.

Over the last 15 years at least, there has been considerable interest in an interface between problems such as these and probability. Vertices are considered to be realizations of some random process, and what is of interest is the behavior (both average and probability one behavior) of such minimal sums as the number of vertices increases.

The remarks below are not intended to be a survey, nor are they intended to be completely rigorous. Our goal is to give the reader a small taste of the appealing nature of some of the problems and solutions often encountered in this area. In particular, we wish to describe how a new boundary rooting technique can be used to prove a rate of convergence result. For the sake of a simple exposition, we will limit our comments to the traveling salesperson problem in dimension 2 where the vertices have the uniform distribution on the unit square. We will only address expected, or average, behavior and leave the statements of probability one results in full generalization (other dimensions, problems, and distributions) to section 5.

2. Two Questions. Let U_1, U_2, U_3, \dots be a sequence of points distributed uniformly and independently in the unit square. Let $L(U_1, U_2, \dots, U_n)$ be the length of a minimal cycle through U_1, U_2, \dots, U_n . (See illustration 1.) $L(U_1, U_2, \dots, U_n)$ is a random variable, and it is natural to inquire about its expectation, $EL(U_1, U_2, \dots, U_n)$, which we will abbreviate EL_n . One question which immediately comes to mind is "How fast does EL_n grow as $n \rightarrow \infty$?" It is not difficult to show that

$$(2.1) \quad C_1 \sqrt{n} \leq EL_n \leq C_2 \sqrt{n}$$

where C_1 and C_2 are positive constants which do not depend upon n . The above display can be rewritten as

$$(2.2) \quad C_1 \leq \frac{EL_n}{\sqrt{n}} \leq C_2.$$

This raises the further problem of whether or not the sequence $\frac{EL_n}{\sqrt{n}}$ converges, which we will call question #1.

Question #1: Does $\frac{EL_n}{\sqrt{n}}$ converge?

This question was answered in 1959 by Beardwood, Halton, and Hammersley.

Theorem #1. *There exists a positive constant α such that $\frac{EL_n}{\sqrt{n}} \rightarrow \alpha$.*

Once a limit theorem such as this has been proved, it makes sense to inquire into the rate of such convergence.

Question #2: How fast does $\left| \frac{EL_n}{\sqrt{n}} - \alpha \right|$ approach zero?

Beardwood, Halton, and Hammersley conjectured that this convergence was at least as fast as $\frac{C}{\sqrt{n}}$. (From this point on, C will represent a constant not depending on n , the value of which may change from line to line.) The truth of this conjecture was not established until 1994 when Alexander proved it.

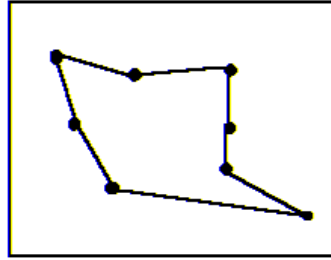
Theorem #2: *There exists a constant C not depending on n such that*

$$\left| \frac{EL_n}{\sqrt{n}} - \alpha \right| \leq \frac{C}{\sqrt{n}}.$$

Essential to the proof of the rate theorem #2 is a property of EL_n called subadditivity. The proof requires establishing upper and lower bounds for the quantity $\alpha - \frac{EL_n}{\sqrt{n}}$. Subadditivity yields the upper bound easily. Most of the work is in establishing the lower bound.

What we intend for the remainder of this note is to give an argument for the lower bound which is different from that of Alexander's. The argument uses a technique called "boundary rooting" which was introduced by Redmond and Yukich (1994). However, it is important for the reader to first understand a bit about subadditivity.

illustration 1



a minimal cycle

3. Subadditivity. Note that if n points are distributed uniformly and independently in a square of edge length $1/m$, then the average length of a minimal cycle through these points is $\frac{1}{m} EL_n$. Now, if n points are distributed uniformly and independently in the unit square which has been subdivided into m^2 subsquares of edge length $1/m$, then “about” $\frac{n}{m^2}$ points fall in a subsquare. (n here is divisible by m^2 .) Therefore, the average length of a minimal cycle through the points falling in a subsquare is “about” $\frac{1}{m} EL_{n/m^2}$. (See illustration 2.)

With some work it is possible to show that the average length of a minimal cycle through all n points is approximately less than the sum of the average lengths of the minimal cycles through the points in each subsquare. Precisely we can say the following.

$$(3.1) \quad EL_n \leq m^2 \left(\frac{1}{m} EL_{n/m^2} \right) + Cm.$$

In the display above, m^2 is the number of subsquares, $\frac{1}{m} EL_{n/m^2}$ is the approximate average length of a minimal cycle through the points in a subsquare, and Cm is an error term accounted for by the fact that there are not exactly n/m^2 points in each subsquare and the fact that the minimal cycles in the subsquares must be patched together before (3.1) can be established.

Now, in (3.1), replace n with nm^2 . The result is the following.

$$(3.2) \quad EL_{nm^2} \leq mEL_n + Cm.$$

This holds for any positive integers n and m . Dividing by $\sqrt{nm^2}$ we obtain

$$(3.3) \quad \frac{EL_{nm^2}}{\sqrt{nm^2}} \leq \frac{EL_n}{\sqrt{n}} + \frac{C}{\sqrt{n}}.$$

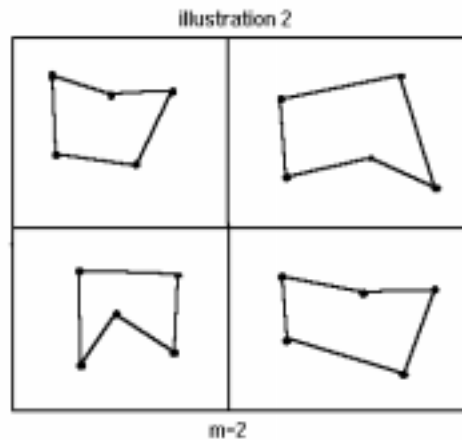
Notice that the right side of this inequality does not depend upon m . Now, if we let $m \rightarrow \infty$, by theorem #1 we obtain

$$\alpha \leq \frac{EL_n}{\sqrt{n}} + \frac{C}{\sqrt{n}}$$

which, rewritten, is

$$(3.4) \quad \alpha - \frac{EL_n}{\sqrt{n}} \leq \frac{C}{\sqrt{n}}.$$

Thus, subadditivity yields easily the upper bound necessary for establishing theorem #2. We now take a different approach to that of Alexander's and show how the boundary rooting technique, in an appealing way, furnishes the necessary lower bound.



4. The Boundary Rooted Traveling Salesperson Tour. Imagine a salesperson traveling along a path in the unit square. Let us define the “cost” of such a path to be the length of that portion of the path which does not intersect the boundary. Hence, whenever the salesperson travels along the boundary, he/she is incurring no cost. Now the salesperson wishes to visit each vertex exactly once and return to the starting point while minimizing the cost of the trip. It may occasionally be more beneficial to go from one vertex to another by exiting to the boundary, traveling the boundary for free, and then reentering to the second vertex. What we now have is a modification of the original traveling salesperson tour, the “boundary rooted” version, where free travel on the boundary is permitted. In this case, let us denote the average length of a minimal boundary rooted cycle through n uniformly and independently distributed points by EL_n .

What is nice about EL_n is that it is superadditive. Imagine n points distributed uniformly and independently in the unit square. Put a minimal boundary rooted cycle through these points. Now subdivide the unit square into m^2 subsquares of edge length $1/m$. One sees that within each subsquare there is a boundary rooted cycle through the points in that subsquare. (See illustrations 4 and 5.) Replace the existing boundary

rooted cycle in each subsquare with a minimal one. This only reduces the cost of the cycle within each subsquare, yielding

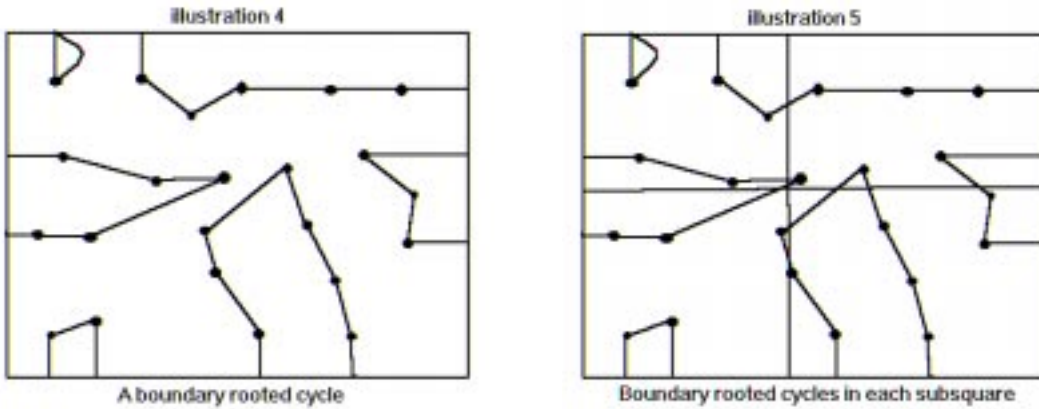
$$(4.1) \quad E\hat{L}_n \geq m^2 \left(\frac{1}{m} E\hat{L}_{n/m^2} \right) - Cm.$$

Again, the error term is due to the fact that there are not exactly n/m^2 points in each subsquare.

The argument in section 3 can now be followed exactly to obtain

$$(4.2) \quad -\frac{C}{\sqrt{n}} \leq \alpha - \frac{E\hat{L}_n}{\sqrt{n}}.$$

(One must first show that $\frac{E\hat{L}_n}{\sqrt{n}} \rightarrow \alpha$, which is not hard to do.) Thus, superadditivity yields the lower bound in the boundary rooted case. What is left to do is to show that $\frac{EL_n}{\sqrt{n}}$ and $\frac{E\hat{L}_n}{\sqrt{n}}$ are sufficiently close to one another.



5. The Relationship between the Original and Boundary Rooted Versions. It is possible to show that

$$(5.1) \quad \left| \frac{EL_n}{\sqrt{n}} - \frac{E\hat{L}_n}{\sqrt{n}} \right| \leq \frac{C}{\sqrt{n}}.$$

From this, (3.4), and (4.2) it follows that

$$(5.2) \quad \alpha - \frac{EL_n}{\sqrt{n}} \geq \alpha - \frac{E\hat{L}_n}{\sqrt{n}} - \frac{C}{\sqrt{n}} \geq -\frac{C}{\sqrt{n}}$$

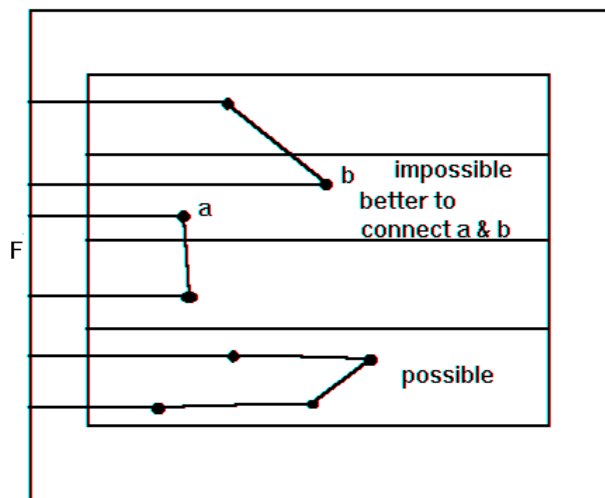
thus finishing the proof of theorem #2.

The key to the proof of (5.1) is to show that in a minimal boundary rooted cycle, the average number of points rooted to the boundary is not more than $C\sqrt{n}$. This is the only part of the argument that we will show.

Let Q denote the unit square, and let Q_n be the square centered within Q with edge length $1 - \frac{2}{\sqrt{n}}$. Between the boundaries of Q and Q_n we have constructed a “mote” of width $\frac{1}{\sqrt{n}}$. The average number of points falling within the mote is not more than $4\sqrt{n}$. Therefore, the average number of points connected to the boundary from within the mote in a minimal boundary rooted cycle is not more than $4\sqrt{n}$.

Now subdivide Q_n into horizontal columns, each of width not more than $\frac{1}{\sqrt{n}}$, so that there are not more than \sqrt{n} of these columns. Let F be one of the vertical sides of Q . Now suppose that a and b are vertices within a column. Note that it is impossible for the minimal boundary rooted cycle to go from a to F , proceed along F , and then enter back to b . Because the mote would be crossed twice, the cost of this part of the cycle would be at least $\frac{1}{\sqrt{n}}$ more than the distance between a and b . This argument shows that there can be at most two points connected to F from within each column. Therefore, the total number of points from within Q_n connected to F is not more than $2\sqrt{n}$ points. Repeating this argument for each of the 4 sides of Q we obtain that the average total number of points connected to the boundary is not more than $C\sqrt{n}$.

illustration 6



6. Some Generalizations. Theorem #1 can be extended and generalized in many different directions. The work of Beardwood, Halton, and Hammersley (1959) along

with that of Steele (1981) shows that if X_1, X_2, \dots is a sequence of points independently distributed in the d -dimensional cube ($d \geq 2$), each with the same distribution, then

$$(6.1) \quad \frac{L(X_1, \dots, X_n)}{n^{\frac{d-1}{d}}} \rightarrow \alpha(L, d) \int f(x)^{\frac{d-1}{d}} dx \quad \text{a.s.}$$

where f is the density of the absolutely continuous part of the probability law of X_1 , a.s. means “almost surely” or “with probability one”, and α is a positive constant depending on d and L . L here is a functional defined on finite sets of points in the d -dimensional cube that satisfies certain conditions. L could be the length of a minimal cycle, but it could also be the length of a minimal spanning tree, a minimal matching, etc.

Theorem #2 can also be extended and generalized. Alexander (1994) and Redmond and Yukich (1994) show that if U_1, U_2, \dots is a sequence of points independently and uniformly distributed in the d -dimensional cube, then

$$(6.2) \quad \left| \frac{EL(U_1, \dots, U_n)}{n^{\frac{d-1}{d}}} - \alpha(L, d) \right| \leq \frac{C}{n^{1/d}}$$

where L again is a functional satisfying certain conditions.

6. Concluding Remarks. Not only has the boundary rooting technique been useful in providing another interesting solution to the Beardwood, Halton, and Hammersley conjecture, but it has helped to unify much of the theory in a natural and simple way. For more on the boundary rooting technique we refer the reader to the Redmond-Yukich and Yukich references. Unfortunately, a recent, general survey of the interface between probability and combinatorial optimization is not yet available. Several are expected in the near future.

Below, in the references, we mention only those articles to which we have referred directly. We are leaving out an immense number of important works in this area and many authors to which this subject owes its present state of development. We hope that what we have listed is enough to point the interested reader in the right direction.

REFERENCES

- ALEXANDER, K. (1994). Rates of convergence of means for distance-minimizing subadditive Euclidean functionals. *Ann. Appl. Probab.* **4** 902-922.
- BEARDWOOD, J., HALTON, J. H. and HAMMERSLEY, J. M. (1959). The shortest path through many points. *Proc. Cambridge Philos. Soc.* **55** 299-327.
- REDMOND, C. and YUKICH, J. E. (1994). Limit theorems and rates of convergence for Euclidean functionals. *Ann. Appl. Probab.* **4** 1057-1073.
- REDMOND, C. and YUKICH, J.E. (1996). Asymptotics for Euclidean functionals with power-weighted edges. *Stochastic Processes and Their Applications.* To appear.
- STEELE, J. M. (1981). Subadditive Euclidean functionals and non-linear growth in geometric probability. *Ann. Probab.* **9** 365-376.

- YUKICH, J. E. (1995). Asymptotics for the stochastic TSP with power-weighted edges. *Prob. Theory and Related Fields*. **102** 203-220.
- YUKICH, J. E. (1995). Quasi-additive Euclidean functionals, in: D. Aldous, P. Diaconis, J. Spencer and J. M. Steele, eds., *Discrete Probability and Algorithms* (Springer, Berlin). 149-158.
- YUKICH, J. E. (1995). Minimal triangulations resemble minimal tours. Preprint.
- YUKICH, J. E. (1995). Worst case growth rates for some classical optimization problems. *Combinatorica*. To appear.
- YUKICH, J. E. (1995). Ergodic theorems for some classical optimization problems. *Ann. Appl. Probab.* To appear.

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