Mathematics Competition

Indiana University of Pennsylvania

2007

Directions:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. If $f(x) = x^2 - 5x + 4$, then

$$f(0) + 2f(1) + 3f(2) + 4f(5/2) + 3f(3) + 2f(4) + f(5)$$

is equal to:

(A) 5 (B) -2 (C) 0 (D) 15 (E) -13

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2. The product of two consecutive even positive integers is 288. The numbers are:

(A) -18, -16 (B) 14, 16 (C) 18, 20 (D) 16, 18 (E) none of these

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3. The value of

$$(\log_2 1/2)(\log_3 9)(\log_4 1/64)(\log_5 625)$$

is:

(A) 1 (B) -2 (C) -6 (D) 24 (E) -120

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4. A man has a 50-foot roll of fencing to make a rectangular shaped kennel. If he wants the kennel to be 6 feet longer than it is wide, the dimensions must be:

(A) 3.5 ft $\times$ 9.5 ft (B) 11 ft $\times$ 17 ft (C) 9.5 ft $\times$ 15.5 ft
(D) 5 ft $\times$ 11 ft (E) none of these

---

5. The area of a rhombus with one side of length 13 and one diagonal of length 10 is:

(A) 100 (B) 120 (C) 140 (D) 160 (E) 180

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6. Of the following polynomials, the one that is a perfect cube of a binomial is:

(A) $8x^3 + 38x^2 + 33x - 27$ (B) $8x^3 + 70x^2 - 21x - 27$
(C) $8x^3 - 86x^2 + 129x - 27$ (D) $8x^3 - 36x^2 + 54x - 27$
(E) none of these

---

7. The expression $\cos(4x + 5y) - \cos(4x - 5y)$ is equivalent to:

(A) $-2\sin 4x \sin 5y$ (B) $2\sin 4x \sin 5y$ (C) $2\cos 4x \cos 5y$
(D) $-2\cos 4x \cos 5y$ (E) $-2\sin 4x \cos 5y$

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8. When $x + 2$ is divided by $x^5 - 3x^3 + 1$, the remainder is:

(A) 9 (B) -7 (C) $x + 2$ (D) $x^4 - 2x^3 + x^2 - 2x + 4$
(E) none of these

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9. A girl dives into a circular pool with a diameter of 100 yards. If she swims straight for 50 yards until she reaches the edge of the pool and then turns right, the remaining distance she will swim until she reaches another point on the edge is:

(A) $50\sqrt{3}$ yds (B) $50\sqrt{2}$ yds (C) 50 yds (D) 90 yds
(E) impossible to determine

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10. Suppose that every person in a room shook hands with every other person in the room exactly once. If the number of handshakes that took place was between 25 and 50 then the set of possible numbers of people in the room is:

(A) $\{9\}$ (B) $\{7, 8, 9\}$ (C) $\{8, 9, 10\}$ (D) $\{7, 8, 9, 10\}$
(E) none of these

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11. The expression

$$\frac{9\sin^2\alpha - 16}{4\sin\alpha - 4} \cdot \frac{\sin^2\alpha - 1}{21\sin\alpha - 28}$$

simplifies to:

(A) $\dfrac{(3\sin\alpha - 4)(\sin\alpha - 1)}{28}$ (B) $\dfrac{(3\sin\alpha + 4)(\sin\alpha + 1)}{28}$

(C) $\dfrac{(3\sin\alpha + 4)(\sin\alpha - 1)}{28}$ (D) $\dfrac{(3\sin\alpha - 4)(\sin\alpha + 1)}{28}$

(E) none of these

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12. The solution set to the equation

$$\frac{\ln(8x) - 3\ln(x)}{\ln(x)} = 1$$

is:

(A) $\{2\}$ (B) $\{0\}$ (C) $\{-2\}$ (D) $\{0, 2\}$ (E) none of these

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13. The solution set to the equation

$$|7x + 12| = |x - 6|$$

is:

(A) $\{-3, -3/4\}$ (B) $\{-3, 3/4\}$ (C) $\{3, -3/4\}$ (D) $\{1, 9/4\}$
(E) none of these

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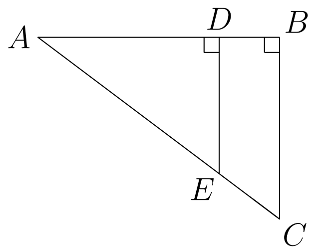
14. In the figure, $DB = 4$, $DE = 9$, and $AD = BC$. The length of AC is:



(A) 16
(B) 18
(C) 20
(D) 22
(E) 24

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15. A man can drive a motorboat 45 miles down the Rock River in the same amount of time that he can drive 27 miles upstream. If the speed of the boat is 12 mph in still water, the speed of the current is:

(A) 15 mph (B) 12 mph (C) 9 mph (D) 6 mph (E) 3 mph

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16. Consider the sequence of binary numbers $(101, 110, 1000, 1100, 10100, \ldots)$. The next number in this sequence is:

(A) 11000 (B) 10000 (C) 100000 (D) 100100 (E) none of these

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17. The set of all solutions to the equation

$$\cos 2x - \cos x = 0$$

in the interval $[0, 2\pi)$ is:

(A) $\left\{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$ (B) $\left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$ (C) $\left\{0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}\right\}$

(D) $\left\{0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$ (E) $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

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18. The solution set to the inequality

$$x^3 - 2x^2 > 4x - 8$$

is:

(A) $(2, \infty)$ (B) $(-2, \infty)$ (C) $(-\infty, -2) \cup (2, \infty)$ (D) $(-2, 2) \cup (2, \infty)$
(E) none of these

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19. There are 64 teams in a double elimination tournament. The least number of games that must be played by all the teams combined in order to determine a champion is:

(A) 64 (B) 126 (C) 128 (D) 255 (E) 256

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20. The largest solution to the equation

$$\sqrt[4]{x^4 + 4x^2 - 4} = -x$$

is:

(A) 1 (B) 2 (C) -2 (D) -1 (E) none of these

---

21. The sides of a triangle are 30, 70, and 80. If an altitude is dropped upon the side of length 80, the length of the larger segment cut off on this side is:

(A) 55 (B) 60 (C) 65 (D) 70 (E) 75

---

22. The solution to the equation

$$\sqrt{\log_{10} \sqrt[4]{x} - \log_{10} \sqrt[5]{x}} = \frac{1}{\sqrt{10}}$$

is:

(A) 0.01 (B) 0.1 (C) 10 (D) 100 (E) 1000

---

23. The number of real solutions to the equation

$$2 + \frac{3}{x^2 + 3x - 10} = \frac{2x^2 - 5}{x^2 + 3x - 10}$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

---

24. The area of the region that lies below the graph of

$$y = \begin{cases} x + 3 & \text{if } x < 1 \\ -2x + 6 & \text{if } x \geq 1 \end{cases}$$

and above the graph of $y = -2$ is:

(A) 9 (B) 18 (C) 27 (D) 36 (E) 45

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25. If

$$\frac{x+y}{x-y} = 12 \quad \text{and} \quad x^2 - y^2 = 48$$

then one possible value for x lies in the interval:

(A) $[-10, -5)$ (B) $[-5, -1)$ (C) $[-1, 10)$ (D) $[10, 20)$ (E) $[20, \infty)$

---

26. The only false statement among the following is:

(A) $1 + 6 + 11 + \cdots + (5n - 4) = \dfrac{n(5n-3)}{2}$

(B) $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n-1)}{2}$

(C) $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

(D) $1 + 5 + 9 + \cdots + (4n - 3) = \dfrac{n(4n-2)}{2}$

(E) $1 + 3 + 5 + \cdots + (2n - 1) = \dfrac{n(2n-1)}{2}$

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27. In the figure, $\angle ABC = 45°$ and $\angle ACB = 120°$. If $AC = 1/2$, then the length of AB is:

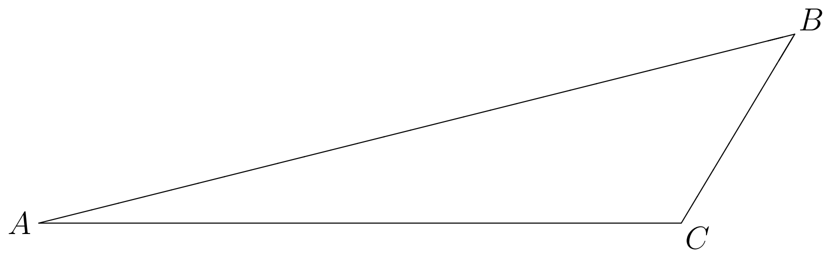
(A) $\sqrt{6}$

(B) $\dfrac{\sqrt{6}}{2}$

(C) $\dfrac{\sqrt{2}}{2}$

(D) $\dfrac{\sqrt{3}}{2}$

(E) $\dfrac{\sqrt{6}}{4}$



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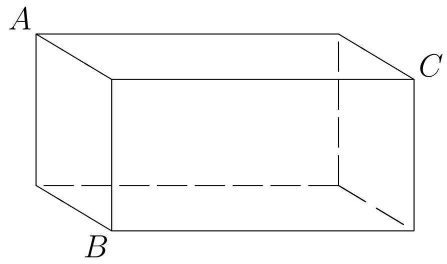
28. A consumer bought three more CDs than DVDs. If a total of \$80 was spent on CDs and \$65 on DVDs and a DVD has a cost per unit that is 30% greater than that for a CD, then the number of DVDs purchased was:

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

---

29. The points A, B, and C are vertices of a rectangular box as shown in the figure below. If $AB = 11$ cm, $AC = 20$ cm, and $BC = 21$ cm, then the volume of the box is:

(A) 360 cm^2
(B) 640 cm^2
(C) 810 cm^2
(D) 960 cm^2
(E) 1080 cm^2



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30. The solution set, in interval notation, to the inequality

$$\frac{(x-1)^2(3-x)}{(x-5)^3} \leq 0$$

is:

(A) $(-\infty, 1) \cup (1, 3) \cup (5, \infty)$ (B) $(-\infty, 3] \cup (5, \infty)$ (C) $(-\infty, 1] \cup (5, \infty)$
(D) $(-\infty, 3] \cup [5, \infty)$ (E) $(-\infty, \infty)$

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31. If you write down the counting numbers from 1 to 1500, the number of times you write down the digit 1 is:

(A) 501 (B) 801 (C) 1001 (D) 1251 (E) 1501

---

32. The solution set to the equation

$$\log_{10}(x-6) - \log_{10}(x-2) = \log_{10}(5/x)$$

is:

(A) $\{-5, 5\}$ (B) $\{1, 10\}$ (C) $\{1, -10\}$ (D) $\{1\}$ (E) $\{10\}$

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33. The number of solutions to the equation

$$|x^2 - 2x - 3| = x + 1$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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34. The area of a circle inscribed in a hexagon is 100π. The area of the hexagon is:

(A) $240\sqrt{2}$ (B) $200\sqrt{3}$ (C) $144\sqrt{6}$ (D) $100\sqrt{12}$ (E) none of these

---

35. The solution to the equation

$$\sqrt{x-3} - \sqrt{x-4} - \sqrt{4x-15} = 0$$

is contained in the interval:

(A) $[3, 5)$ (B) $[5, 8]$ (C) $(8, 10]$ (D) $(10, \infty)$ (E) none of these

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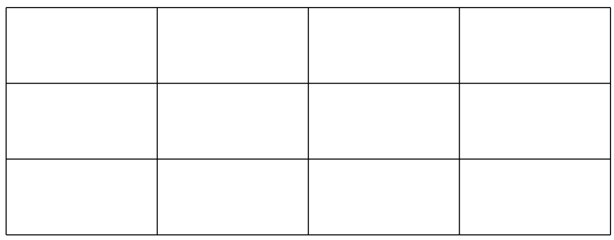
36. The number of rectangles of any size in the figure below is:

(A) 60
(B) 45
(C) 36
(D) 20
(E) none of these



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37. The number of solutions to the equation

$$8\sin^2 x \cos^2 x - 4\sin^2 x - 6\cos^2 x + 3 = 0$$

in the interval $[0, \pi]$ is:

(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

---

38. If $x^2 - xy = 48$ and $xy - y^2 = 12$, the largest possible value of y is:

(A) 2 (B) 6 (C) 8 (D) 14 (E) none of these

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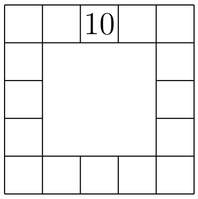
39. The triangular numbers are $1, 3, 6, 10, 15, 21, 28, \ldots$. Starting with 10 in the box as shown, place the rest of the counting numbers from 1 to 16 into the sixteen boxes so that for any two boxes that share an edge the sum of the numbers is a triangular number. The sum of the four corners is:



(A) 19
(B) 27
(C) 34
(D) 48
(E) 49

---

40. The solution set to the inequality

$$5 < x + 2 < 2x + 1$$

is:

(A) $(1, \infty)$ (B) $(3, \infty)$ (C) $(1, 3)$ (D) $(-\infty, 1) \cup (3, \infty)$
(E) none of these

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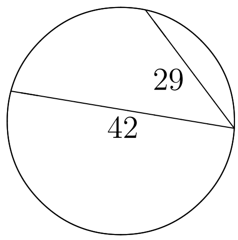
41. The chord of a minor arc of a circle is 42 and the chord of one half the same arc is 29, as shown. The radius of the circle is:



(A) 21.025
(B) 20
(C) 10
(D) 40
(E) $\sqrt{923}$

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42. The diameter of circle A is 6 units and the diameter of circle B is 12 units. If the area of circle A is increasing at a rate of 20% per minute and the area of circle B is increasing at a rate of 10% per minute, the number of minutes until the areas will be equal is:

(A) $\dfrac{\ln(2)}{\ln(1.1) - \ln(1.2)}$ (B) $\dfrac{\ln(2)}{\ln(1.2) - \ln(1.1)}$ (C) $\dfrac{1}{\ln(1.2) - 2\ln(1.1)}$

(D) $\dfrac{2}{\ln(1.2) - \ln(1.1)}$ (E) $\dfrac{\ln(4)}{\ln(1.2) - \ln(1.1)}$

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43. Assuming $a \neq 0$, $b \neq 0$, $a \neq \pm 2b$, $2a \neq \pm 3b$, and $3a \neq \pm 4b$, the sum of the solutions to the equation

$$\frac{1}{2x^2+x-1}+\frac{1}{2x^2-3x+1}=\frac{a}{2bx-b}-\frac{2bx+b}{ax^2-a}$$

is:

(A) $\dfrac{a+b}{2b+a}$ (B) $\dfrac{b-a}{a-2b}$ (C) $\dfrac{a-b}{a^2-4b^2}$ (D) $\dfrac{2ab}{a^2-4b^2}$ (E) $\dfrac{a+b}{a^2-4b^2}$

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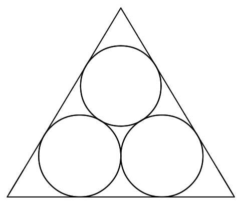
44. Sally, Bob, and Jimmy can run one lap around a track in 30 seconds, 50 seconds, and 70 seconds, respectively. If they all begin running from the starting line, the time it will take for all three of them to be together again next is:

(A) 150 sec (B) 210 sec (C) 525 sec (D) 1050 sec (E) never

---

45. Three circles of radius 1 are packed into an equilateral triangle, as shown. The area of the triangle is:

(A) $3+2\sqrt{3}$
(B) $6+4\sqrt{3}$
(C) $4\sqrt{3}$
(D) 3π
(E) $4\sqrt{3}-3\pi$



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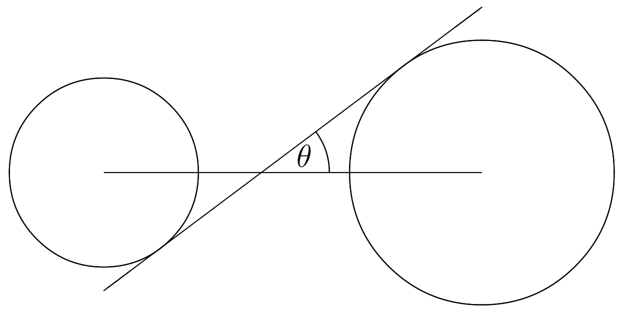
46. A train travels at its usual rate for one hour and is then detained for 15 minutes, after which it proceeds at 3/4 its former rate and arrives 24 minutes late. If the detention had occurred 5 miles further on, the train would have been only 21 minutes late. The usual rate of the train is:

(A) $30\frac{1}{3}$ mph (B) $33\frac{1}{3}$ mph (C) $35\frac{2}{3}$ mph (D) $36\frac{2}{3}$ mph
(E) none of these

---

47. Two circles with radii 1.25 cm and 1.75 cm, respectively, are drawn with 5 cm between their centers. A line is tangent to both. Angle θ is formed by the tangent line and the line between their centers. Then $\tan\theta$ is equal to:



(A) $\frac{5}{3}$

(B) $\frac{5}{7}$

(C) $\frac{3}{4}$

(D) $\frac{7}{20}$

(E) $\frac{3}{5}$

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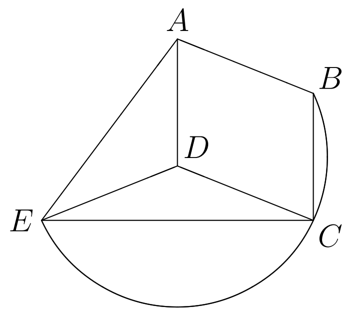
48. In the figure, $ABCD$ is a parallelogram, CDE is an isosceles triangle, and $\angle BCE$ is inscribed in a semicircle. If $AD = 11$, $DE = 13$, and $AE = 20$, then the area of the quadrilateral $ABCE$ is:



(A) 264
(B) 196
(C) 316
(D) 212
(E) 258

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49. The sum of the real solutions to the equation

$$\frac{x+\sqrt{x^2-9}}{x-\sqrt{x^2-9}} = x-2$$

is:

(A) 0 (B) 1 (C) $1+\sqrt{3}$ (D) $2-\sqrt{3}$ (E) 3

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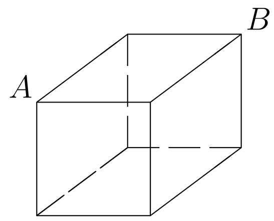
50. The number of ways to travel from vertex A to vertex B by moving along the edges of the cube and visiting each vertex no more than once is:



(A) 16
(B) 18
(C) 20
(D) 22
(E) 24

Answer Key

1. E
2. D
3. D
4. C
5. B
6. D
7. A
8. C
9. A
10. C
11. B
12. A
13. A
14. C
15. E
16. D
17. B
18. D
19. B
20. D
21. C
22. D
23. A
24. C
25. D
26. E
27. E
28. C
29. E
30. B
31. C
32. E
33. D
34. B
35. A
36. A
37. B
38. A
39. E
40. B
41. A
42. E
43. D
44. C
45. B
46. B
47. C
48. E
49. E
50. A