Mathematics Competition

Indiana University of Pennsylvania

2000

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. Successive discounts of 10% and 15% are equivalent to a single discount of:

(A) 12.5% (B) 23.5% (C) 25% (D) 76.5% (E) none of these

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2. If a 75 m wire is cut into three pieces having proportions 2:4:6, then the length, in meters, of the smallest piece is:

(A) $6\frac{1}{4}$ (B) $12\frac{1}{2}$ (C) $13\frac{1}{3}$ (D) $16\frac{2}{3}$ (E) none of these

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3. A 25 ft ladder is placed against a vertical wall. The foot of the ladder is 7 feet from the base of the wall. If the foot of the ladder slips 8 feet, then the top of the ladder will slide:

(A) 4 ft (B) 6 ft (C) 8 ft (D) 20 ft (E) none of these

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4. The sum of the roots of the equation

$$(x^2 + 5x - 1)(x - 4) = 0$$

is:

(A) -5 (B) 1 (C) -1 (D) -9 (E) none of these

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5. If $b > 1$, then $\log_b 2b^2 - \log_b 2b^6$ is equal to:

(A) -4 (B) -8 (C) $-4(1 + \log_b 2)$ (D) $2 + \log_b 2 + \log_b(1 - b^4)$
(E) none of these

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6. An IUP runner has reduced her time for the Indiana-Nowhere Run from 5 hr to 4 hr. She also has increased her speed by 2 mph. The distance of the Indiana-Nowhere Run is:

(A) 8 mi (B) 16 mi (C) 20 mi (D) 32 mi (E) none of these

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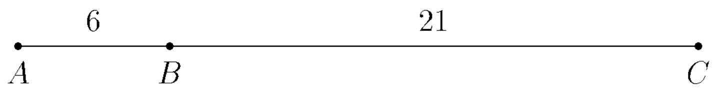
7. Given the line segment



the smallest number of equal segments into which $\overline{AC}$ can be divided so that the fixed point B is the endpoint of two segments is:

(A) 2 (B) 4 (C) 9 (D) 27 (E) none of these

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8. The number $\sqrt{2}+\sqrt{3}$ is a root of:

(A) x^4-10x^2+1 (B) x^4-10x^2+2 (C) x^4-10x^2+3
(D) x^4-10x^2+6 (E) x^4-10x^2+10

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9. The period of the function $f(t)=5\sin \pi t$ is:

(A) 5 (B) $2\pi/5$ (C) 2 (D) π (E) 2π

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10. Consider a $\triangle ABC$ where $\angle A$ has measure $75°$ and $\angle B$ has measure $45°$. If the length of $\overline{AC}$ is $\sqrt{2}$, then the length of $\overline{AB}$ is:

(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) $\dfrac{2}{\sqrt{3}}$ (D) $\dfrac{\sqrt{2}}{3}$ (E) none of these

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11. The expression $(x^{-1}-y^{-1})^{-1}$ is equal to:

(A) $\dfrac{xy}{x-y}$ (B) $x+y$ (C) $x-y$ (D) $\dfrac{y-x}{xy}$ (E) none of these

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12. Suppose you have the five Scrabble tiles M, S, E, O, and O. If you randomly place the tiles in a row, the probability that you spell the word MOOSE is:

(A) 1/5 (B) 1/10 (C) 1/60 (D) 1/120 (E) none of these

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13. The positive real root of

$$x^4 + 8x^2 - 5 = 0$$

is:

(A) $\sqrt{-4+\sqrt{21}}$ (B) $\sqrt{4+\sqrt{21}}$ (C) $\sqrt{-4+\sqrt{17}}$ (D) $\frac{\sqrt{21}}{5}$
(E) $\frac{\sqrt{-4+\sqrt{21}}}{2}$

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14. If the graph of $\ln y$ versus x is linear but not constant, then the graph of y versus x is:

(A) constant (B) linear but not constant (C) logarithmic
(D) exponential (E) none of these

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15. If $x/y = 3/7$, then the incorrect expression among the following is:

(A) $\frac{x+y}{y} = \frac{10}{7}$ (B) $\frac{y}{y-x} = \frac{7}{4}$ (C) $\frac{x+2y}{2y} = \frac{3}{7}$ (D) $\frac{x}{3y} = \frac{1}{7}$

(E) $\frac{x-y}{x} = \frac{-4}{3}$

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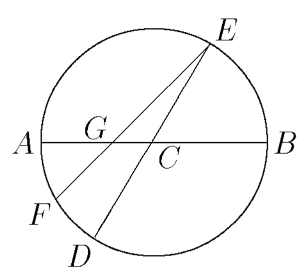
16. In the figure, $\overline{AB}$ and $\overline{DE}$ are diameters of the given circle, intersecting on the center C of the circle. Also, F is the midpoint of the minor arc determined by points A and D, and the chord $\overline{EF}$ intersects $\overline{AB}$ on the point G. If $\angle BCE$ has measure $60°$, then the measure of $\angle AGF$ is:

(A) $45°$
(B) $30°$
(C) $15°$
(D) impossible to determine
(E) none of these



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17. The value of $\ln\left(\ln\left(\ln\left(\ln\left(\ln e^{e^{\sqrt{e}}}\right)\right)\right)\right)$ is:

(A) e (B) $1/2$ (C) $\ln(1/2)$ (D) $-1/2$ (E) not a real number

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18. The number of real solutions of the equation

$$e^{2x^2} + (x^2+1)e^{x^2} + x^2 = 0$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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19. The side of a square is 3 inches longer than the side of an equilateral triangle. The perimeter of the square is twice the perimeter of the triangle. The area of the triangle in square inches is:

(A) $\frac{9\sqrt{3}}{8}$ (B) $\frac{9\sqrt{3}}{2}$ (C) $18\sqrt{3}$ (D) $9\sqrt{3}$ (E) none of these

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20. A man makes a trip by automobile at an average speed of 50 mph. He returns over the same route at an average speed of 40 mph. His average speed for the entire trip is:

(A) $43\frac{7}{9}$ mph (B) $44\frac{4}{9}$ mph (C) 45 mph (D) $45\frac{1}{9}$ mph
(E) none of these

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21. Let $f(x) = \dfrac{x}{x+3}$. The number of values of x_0 for which $f(f(x_0)) = x_0$ but $f(x_0) \neq x_0$ is:

(A) 0 (B) 1 (C) 2 (D) infinite (E) none of these

---

22. The radius of the circle passing through the points $(-10, 9)$, $(4, 11)$, and $(6, -3)$ is:

(A) 7 (B) 10 (C) $10\sqrt{2}$ (D) 20 (E) none of these

---

23. The solution set, in interval notation, of the inequality

$$\frac{x^2(x-2)^3}{x^2-1} \leq 0$$

is:

(A) $(-\infty, -1) \cup [0, 0] \cup (1, 2]$ (B) $(-1, 1) \cup [2, \infty)$ (C) $(-\infty, -1] \cup [1, 2]$
(D) $[-1, 1] \cup (2, \infty)$ (E) $(-1, 2]$

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24. The domain of the function $f(x) = \dfrac{\sqrt{5-x}}{\sqrt[4]{x^2-5x+4}}$ in interval notation is:

(A) $(-\infty, 1] \cup [4, 5]$ (B) $(-\infty, 1) \cup (4, \infty)$ (C) $(1, 5]$
(D) $(-\infty, 1) \cup (4, 5]$ (E) $(1, 4)$

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25. If $x^3 = 1 + x + x^2$, then x^8 is equal to:

(A) $1 + 2x + 2x^2$ (B) $2 + 3x + 4x^2$ (C) $4 + 6x + 7x^2$
(D) $7 + 11x + 13x^2$ (E) $13 + 20x + 24x^2$

---

26. A circle of radius 3 is inscribed in an arbitrary triangle as shown in the figure. The circle is tangent to side $\overline{AB}$ at the point P. If $AP = 5$ and $PB = 4$, then the sine of the angle A is:

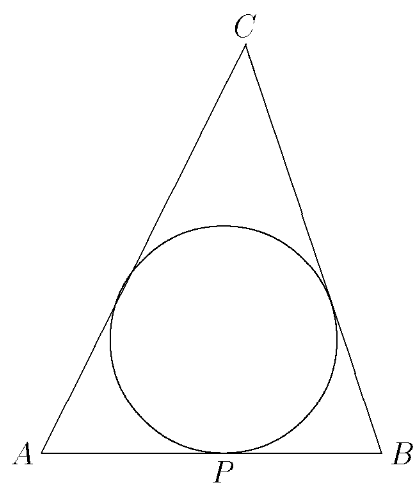
(A) $\dfrac{3}{5}$

(B) $\dfrac{4}{5}$

(C) $\dfrac{15}{17}$

(D) $\dfrac{3}{\sqrt{34}}$

(E) none of these



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27. Of all the 4-digit numbers that can be formed using digits 1 through 9, the quantity of numbers in which the digit 3 occurs at least once is:

(A) 2048 (B) 2465 (C) 2879 (D) 6561 (E) none of these

---

28. The number of integer roots of

$$x^{2/3} - 5 + 4x^{-2/3} = 0$$

is:

(A) 0 (B) 1 (C) 2 (D) 4 (E) none of these

---

29. The solution of the equation

$$e^{x\ln 2}e^{x\ln 3}e^{x\ln 5}e^{x\ln 7} = 9{,}261{,}000$$

is:

(A) 210 (B) e^{210} (C) $\ln 210$ (D) 3 (E) $\ln 3$

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30. The positive value of y that satisfies the system of equations

$$\begin{aligned} x - 3 &= xw, \\ y - 1 &= yw, \\ z + 1 &= zw, \\ x^2 + y^2 + z^2 &= 4 \end{aligned}$$

is equal to:

(A) $1/\sqrt{11}$ (B) $2/\sqrt{11}$ (C) $3/\sqrt{11}$ (D) $4/\sqrt{11}$ (E) $5/\sqrt{11}$

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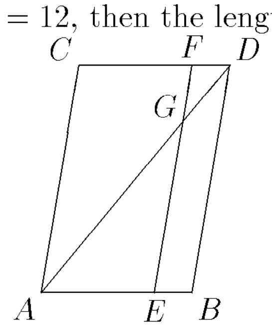
31. In the figure, $\overline{AB}$ is parallel to $\overline{CD}$ and $\overline{AC}$, $\overline{BD}$, and $\overline{EF}$ are parallel. The lines $\overline{EF}$ and $\overline{AD}$ intersect at G. If $FG = 3$, $EB = 2$, and $BD = 12$, then the length of $\overline{AE}$ is:



(A) 4
(B) 6
(C) 8
(D) 10
(E) none of these

---

32. The number of solutions to

$$2\cos^2 2x - 2\sin x\cos x - 1 = 0$$

in $[0, 2\pi]$ is:

(A) 0 (B) 2 (C) 3 (D) 4 (E) 6

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33. The system of equations

$$\begin{aligned} x^2 + ax + 1 &= 0 \\ x^2 + x + a &= 0 \end{aligned}$$

has at least one root (real or complex) for:

(A) $a = 1$ only (B) $a = -2$ only (C) $a = 1$ and $a = -2$ only
(D) all values of a (E) none of these

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34. If $f(x) = 2x - 4$ and $g(x) = 3x + 5$, then the only false statement among the following is:

(A) $f^{-1}(x) = \dfrac{x+4}{2}$ (B) $g^{-1}(x) = \dfrac{x-5}{3}$

(C) Domain(f) = Range(f^{-1}) (D) $(f \circ g)^{-1}(x) = (f^{-1} \circ g^{-1})(x)$

(E) $(f^{-1} \circ f \circ g)(x) = 3x + 5$

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35. When $x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$ is divided by $x^2 - 1$, the remainder is:

(A) 1 (B) 6 (C) -6 (D) $6x$ (E) none of these

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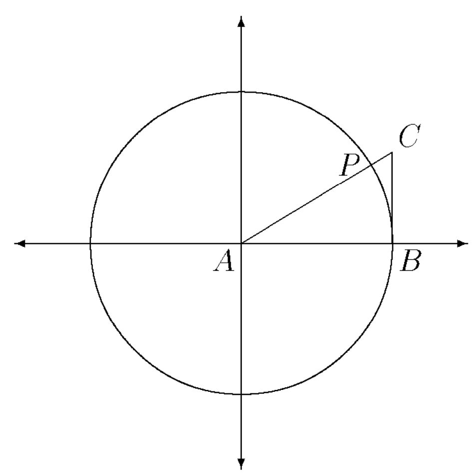
36. In the figure, $\overline{BC}$ is tangent to the circle centered at the origin A. If $\overline{AC}$ intersects the circle at $P(\sqrt{3}/2, 1/2)$, then the length of $\overline{PC}$ is:



(A) $\dfrac{2}{\sqrt{3}}$

(B) $\sqrt{3} - 1$

(C) $\dfrac{2\sqrt{3} - 3}{3}$

(D) 1

(E) $\dfrac{3}{2}$

---

37. If $f(2x+3) = \dfrac{4}{6x-1}$, then $f(x)$ equals:

(A) $\dfrac{5-18x}{6x-1}$ (B) $\dfrac{4}{3x-8}$ (C) $\dfrac{18x+5}{6x-1}$ (D) $\dfrac{4}{3x-10}$

(E) none of these

---

38. The solution set, in interval notation, of the inequality

$$0 < |2x-4| < |2x-8|$$

is:

(A) $(2,4) \cup (4,\infty)$ (B) $(-\infty,\infty)$ (C) $(-\infty,2) \cup (2,3)$ (D) $(-\infty,3)$
(E) $(3,\infty)$

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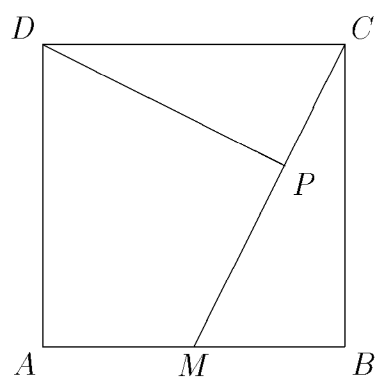
39. In the accompanying figure, $ABCD$ is a square, each side of which has length 1. If M is the midpoint of AB and $DP \perp MC$, then DP has length:



(A) $\dfrac{\sqrt{5}}{2}$

(B) $\dfrac{2\sqrt{5}}{5}$

(C) $\dfrac{\sqrt{3}}{2}$

(D) 1

(E) none of these

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40. Bigger Hammer Hardware Store is currently selling widgets at \$10 apiece. On average, 215 widgets per month are sold at this price. The managers have found that each dollar increase in price reduces the monthly sales by 25 widgets. If they purchase widgets for the wholesale price of \$7 each, the price they must charge for widgets in order to maximize their profit is:

(A) \$9.30 (B) \$13.00 (C) \$18.60 (D) \$25.20 (E) none of these

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41. Consider the function defined by

$$f(x) = \ln\left(\frac{1+x}{1-x}\right).$$

If a and b are numbers in the domain of f with $a \neq \pm b$ and $f(x) = f(a) + f(b)$, then x is:

(A) $\frac{a-b}{1+ab}$ (B) $\frac{1+ab}{a-b}$ (C) $\frac{a+b}{1+ab}$ (D) $\frac{1+ab}{a+b}$ (E) none of these

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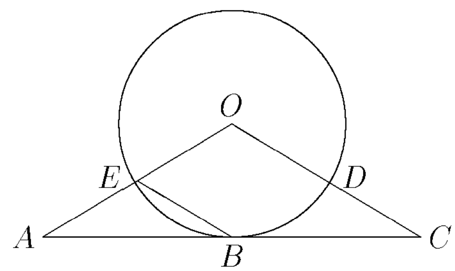
42. In the figure, the circle with center O is tangent to line $\overline{AC}$ at point B. The length of the segment $\overline{AB}$ is $\sqrt{27}$ units and $m(\angle EAB) = m(\angle EBA) = m(\angle BCD)$. If the area of the sector of the circle that is inside $\triangle OAC$ is x square units, then:

(A) $x \leq 2\pi$
(B) $2\pi < x \leq 3\pi$
(C) $3\pi < x \leq 4\pi$
(D) $4\pi < x \leq 5\pi$
(E) $5\pi < x$



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43. The number of distinct real roots of the equation

$$x^4 - \tfrac{11}{4}x^3 - \tfrac{11}{2}x^2 + \tfrac{5}{4}x + 3 = 0$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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44. We wish to determine the height of a tree on an island in a stream. Lying on the shore of the stream, we note that our line of sight to the top of the tree makes an angle of $75°$ with the surface of the water. Backing 20 ft further away from the tree we find the new angle to be $45°$. The height of the tree to the nearest foot is:

(A) 27 ft (B) 28 ft (C) 30 ft (D) 40 ft (E) none of these

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45. A standard deck of cards contains 52 cards, 4 of which are aces. A pinochle deck contains 48 cards, 8 of which are aces. A pinochle deck is in a drawer with three standard decks. A deck is grabbed at random and one card is dealt from that deck. If the card that is dealt is an ace, then the probability that the chosen deck is the pinochle deck is:

(A) 1/4 (B) 13/19 (C) 13/31 (D) 19/52 (E) none of these

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46. The solution set to the equation $2^{2x+1} + 8 = 2^{x+4} + 2^x$ is:

(A) $\{0\}$ (B) $\{3\}$ (C) $\mathbb{R}$ (D) $\emptyset$ (E) none of these

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47. Three tangent circles of radius r are drawn all having centers on the line $\overline{OE}$. Line $\overline{OD}$ is tangent to the right-hand circle at D and intersects the left-hand circle at O. The length of the chord $\overline{AB}$ is:

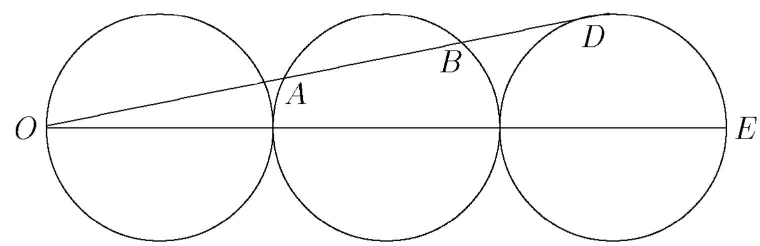
(A) $\dfrac{(1+\sqrt{5})r}{2}$

(B) $\dfrac{2r\sqrt{6}}{3}$

(C) $\dfrac{13r}{8}$

(D) $\dfrac{r\sqrt{65}}{5}$

(E) $\dfrac{8r}{5}$



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48. In the addition problem below, each letter represents a digit in such a way that every digit from 0 to 9 is used once.

$$\begin{array}{r} a\,b\,4 \\ +\,2\,8\,c \\ \hline 1\,d\,e\,f \end{array}$$

The letter c represents:

(A) 3 (B) 5 (C) 6 (D) 9 (E) none of these

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49. The sum of the y-coordinates of the points of intersection of the graphs of the functions $f(x) = 2x^5 + 5x^4 - 8x^3$ and $g(x) = 14x^2 - 6x - 9$ is:

(A) 0 (B) 146 (C) 145 (D) 23.5 (E) 158.5

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50. The expression

$$(1 + 2\cos x + 2\cos 2x + 2\cos 3x + 2\cos 4x)\sin(x/2)$$

simplifies to:

(A) $\sin^2(x/2)$ (B) $2\sin(x/2)\cos(x/2)$ (C) $\cos^2(x/2)$ (D) $\sin(9x/2)$
(E) $\cos(9x/2)$

Answer Key

1. B
2. B
3. A
4. C
5. A
6. E
7. C
8. A
9. C
10. B
11. E
12. C
13. A
14. D
15. C
16. A
17. E
18. A
19. D
20. B
21. A
22. B
23. A
24. D
25. E
26. C
27. B
28. D
29. D
30. B
31. B
32. E
33. C
34. D
35. D
36. C
37. D
38. C
39. B
40. E
41. C
42. B
43. D
44. A
45. C
46. E
47. E
48. D
49. E
50. D