Mathematics Competition
Indiana University of Pennsylvania
2001

Directions:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. A rectangular gold bar that measures $2'' \times 3'' \times 4''$ is melted down and three equal cubes (of maximum size) are constructed from this gold. The length of the side of each cube, in inches, is:

(A) 2 (B) 3 (C) $\sqrt[3]{3}$ (D) $\sqrt{8}$ (E) none of these

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2. A value of k such that the equation $9x^2 = kx - 4$ will have exactly one solution is:

(A) -4 (B) 9 (C) 3 (D) 12 (E) none of these

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3. Let $f(S)$ be the number of gallons of paint one needs to buy to cover a house of surface area S ft^2, assuming one cannot buy fractions of a gallon. Thus if it would take 4.1 gallons of paint to cover the house, then $f(S) = 5$. The expression that matches the following situation—I bought enough paint to cover this house and a welcome sign that measures 2 square feet—is:

(A) $2f(S)$ (B) $f(S) + 2$ (C) $f(S) + f(2)$ (D) $f(S + 2)$
(E) none of these

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4. The sum of the roots of the equation

$$(x^2 - 3x + 1)(x^2 - 4)(x + 3) = 0$$

is equal to:

(A) 0 (B) -6 (C) 5 (D) -5 (E) none of these

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5. If $\log_b 9 = 1/2$, then b is equal to

(A) 3 (B) 4.5 (C) 18 (D) 81 (E) none of these

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6. The minimum value of $2x^2 - 12x + 20$ is:

(A) 4 (B) 3 (C) 2 (D) 1 (E) none of these

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7. Sales of cassette tapes of music decreased by 7% per year for 3 years. The percent that best approximates the total percent that sales of cassette tapes of music decreased during this time period is:

(A) 80.4% (B) 79% (C) 34.3% (D) 21% (E) 19.6%

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8. The number $2\sqrt{6}$ does not equal:

(A) $\sqrt{6}+\sqrt{6}$ (B) $\sqrt{3}\sqrt{8}$ (C) $2\sqrt{3}+2\sqrt{3}$ (D) $\dfrac{12}{\sqrt{6}}$ (E) any of these

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9. If $\triangle ABC$ is a right triangle with legs $AC = 5$ and $BC = 3$, an expression which is not equal to the measure of $\angle A$ is:

(A) $\cos^{-1}\left(\dfrac{5}{\sqrt{34}}\right)$ (B) $\tan^{-1}\left(\dfrac{3}{5}\right)$ (C) $\sin^{-1}\left(\dfrac{3}{4}\right)$ (D) $\cot^{-1}\left(\dfrac{5}{3}\right)$
(E) none of these

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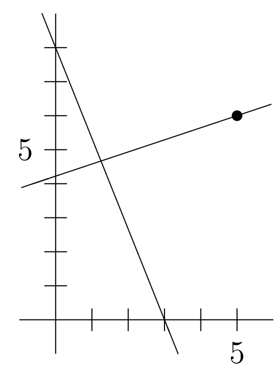
10. If the lines in the figure are perpendicular, then an equation for the line going through the point $(5, 6)$ is:

(A) $y = \dfrac{63}{8} - \dfrac{3}{8}x$
(B) $y = \dfrac{33}{8} + \dfrac{3}{8}x$
(C) $y = \dfrac{58}{3} - \dfrac{8}{3}x$
(D) $y = \dfrac{8}{3}x - \dfrac{22}{3}$
(E) none of these



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11. Of the following lists of fractions, the one list in increasing order is:

(A) $\dfrac{13}{19}, \dfrac{11}{15}, \dfrac{15}{21}$ (B) $\dfrac{13}{19}, \dfrac{15}{21}, \dfrac{11}{15}$ (C) $\dfrac{15}{21}, \dfrac{13}{19}, \dfrac{11}{15}$ (D) $\dfrac{15}{21}, \dfrac{11}{15}, \dfrac{13}{19}$
(E) none of these

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12. The sum of the digits of the least common multiple of 18, 24, and 30 is:

(A) 9 (B) 12 (C) 15 (D) 18 (E) none of these

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13. The expression $\sqrt{25+25x^2} - \sqrt{16+16x^2}$ is equal to:

(A) $1+x^2$ (B) $1+x$ (C) $3\sqrt{1+x^2}$ (D) $9\sqrt{1+x^2}$
(E) none of these

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14. The numerical value of $9^{3\log_3 2}$ is equal to:

(A) 9 (B) 32 (C) 64 (D) 81 (E) none of these

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15. The **harmonic mean** of two numbers x and y is defined by the expression

$$\frac{2}{x^{-1}+y^{-1}}.$$

If the harmonic mean of two numbers is 3 and one of the numbers is 2, then the other number is:

(A) 6 (B) 4 (C) 1 (D) -1 (E) none of these

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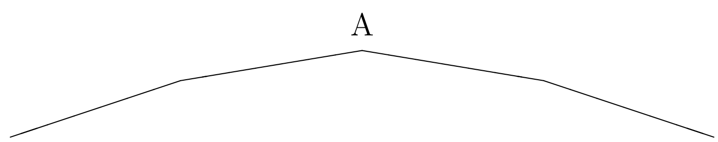
16. Part of the boundary of a regular polygon is shown in the figure below. If $\angle A = 188°$, then the number of sides that this polygon has is:

(A) 45
(B) 60
(C) 90
(D) 18
(E) none of these



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17. If $\tan\theta = 2/3$ and $\cos\theta < 0$, then $\sec\theta$ is equal to:

(A) $\frac{3}{2}$ (B) $\frac{\sqrt{13}}{3}$ (C) $\frac{-\sqrt{13}}{3}$ (D) $\frac{3\sqrt{13}}{13}$ (E) $\frac{-3\sqrt{13}}{13}$

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18. If a and b are not both zero, the solution (x, y) to the system

$$\begin{aligned} ax - by &= 1 \\ bx + ay &= 0 \end{aligned}$$

has:

(A) $x = \frac{a}{a^2-b^2}$ (B) $x = \frac{b}{a^2+b^2}$ (C) $y = \frac{b}{a^2+b^2}$ (D) $y = \frac{-b}{a^2+b^2}$
(E) none of these

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19. If the radius of a right circular cylinder (can) is increased by 20% and the height is decreased by 20%, then the volume is:

(A) unchanged (B) decreased by $\pi\%$ (C) increased by $2\pi\%$
(D) increased by 15.2% (E) none of these

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20. The numbers a, b, and c are proportional to 2, 3, and 5, respectively, and the sum of a, b, and c is 90. If the number b is given by the equation $b = xc - 18$, then x is:

(A) 1/2 (B) 2/3 (C) 3/2 (D) 1 (E) none of these

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21. Three students enter into a partnership. The first puts in \$58, the second \$87. The profit from the partnership is \$368, of which the first student gets \$86. The amount of the profits that should be given to the third student is:

(A) \$129 (B) $\$103\frac{8}{43}$ (C) \$153 (D) \$282 (E) none of these

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22. If the quantity Q is given by

$$Q = Q_0 \left(\frac{1}{16}\right)^{-t/36},$$

where t is measured in hours, then the initial amount will double in:

(A) 36 hours (B) 9 hours (C) 1.5 hours (D) 576 hours
(E) none of these

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23. If reduced to lowest terms, the expression

$$\frac{x^2 + y^2}{xy} + \frac{xy - x^2}{xy - y^2}$$

is equal to:

(A) $\frac{y}{x}$ (B) $\frac{2x^2 + y^2}{xy}$ (C) $x - 2y$ (D) $\frac{y - x}{y}$ (E) none of these

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24. In a small Florida precinct, a ballot box was opened and found to contain eight ballots equally split between Mr. Bush and Mr. Gore. If a later random sample of size four is chosen from these ballots, then the probability that this sample contains two votes for Bush and two votes for Gore is:

(A) 0 (B) $\frac{70}{256}$ (C) $\frac{1}{2}$ (D) $\frac{144}{1680}$ (E) $\frac{18}{35}$

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25. The solution, in interval notation, to the inequality

$$\frac{x-5}{\sqrt{x+2}} \leq 0$$

is:

(A) $(-\infty, 5]$ (B) $(-2, \infty)$ (C) $(-2, 5)$ (D) $[-2, 5]$ (E) $(-2, 5]$

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26. The graph of $y = 7 - x^2$ and the graph of $y = 7 - x$ meet in two points. The distance between these points is:

(A) less than 1 (B) 1 (C) $\sqrt{2}$ (D) 2 (E) greater than 2

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27. Consider the function defined by

$$f(x) = 3x^2 + 5x - 1;\ x \leq \frac{-5}{6}.$$

The inverse function for f is defined by:

(A) $f^{-1} = \frac{1}{6}\sqrt{12x+37} - \frac{5}{6};\ x \geq \frac{-37}{12}$ (B) $f^{-1} = \frac{-1}{6}\sqrt{12x+37} - \frac{5}{6};\ x \geq \frac{-37}{12}$

(C) $f^{-1} = \frac{1}{6}\sqrt{12x+37} + \frac{5}{6};\ x \leq \frac{-5}{6}$ (D) $f^{-1} = \frac{-1}{6}\sqrt{12x+37} - \frac{5}{6};\ x \leq \frac{-5}{6}$

(E) none of these

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28. The sum of the real values of x satisfying the equation $|x+3| = 2|2-x|$ is:

(A) $7\frac{1}{3}$ (B) $\frac{8}{3}$ (C) $\frac{8}{7}$ (D) $\frac{5}{6}$ (E) none of these

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29. For every $x > 0$, the constant c such that

$$\tan^{-1}(x) + \tan^{-1}(1/x) = c$$

is:

(A) $\frac{\pi}{2}$ (B) π (C) 1 (D) 0 (E) none of these

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30. The number of real solutions to the equation

$$\sqrt{x+2}+\sqrt[3]{x-1}=3$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 6

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31. In the figure, A and B are points on the circle centered at O. If the radius of the circle is 2 units and $\angle AOB = 60^\circ$, then the area of the shaded region is:

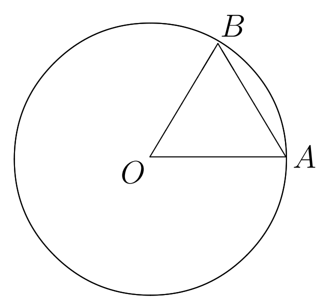
(A) $\dfrac{2\pi}{3} - \sqrt{3}$ sq. units

(B) $\dfrac{2\pi}{3}$ sq. units

(C) $\dfrac{2\pi}{3} + \sqrt{3}$ sq. units

(D) 2 sq. units

(E) $\dfrac{2\pi}{3} - \sqrt{2}$ sq. units



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32. If

$$\frac{3^x+3^{-x}}{2}=4,$$

then x is equal to:

(A) 0 (B) ± 1 (C) $4 \pm \sqrt{15}$ (D) $\ln(4 \pm \sqrt{15})$ (E) $\log_3(4 \pm \sqrt{15})$

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33. The solution, in interval notation, of the inequality

$$\frac{1}{|x+7|} > 2$$

is:

(A) $(-\infty,-7)\cup(-7,\infty)$
(B) $(-15/2,-13/2)$
(C) $(-15/2,-7)$
(D) $(-\infty,-15/2)\cup(-13/2,\infty)$
(E) $(-15/2,-7)\cup(-7,-13/2)$

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34. The graph of the equation

$$x^2y - y^3 - 5x^2 + 5y^2 = 0$$

consists of:

(A) one circle and one line
(B) one parabola and one line
(C) two lines and one ellipse
(D) three lines
(E) two parabolas

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35. If x is a real number, then $\lfloor x \rfloor$ is defined to be the greatest integer less than or equal to x. The solution, in interval notation, of the equation

$$2\lfloor 1 - x \rfloor - 1 = 5$$

is:

(A) $(-4, -3]$ (B) $(-3, -2]$ (C) $(-2, -1]$ (D) $(1, 2]$ (E) $(2, 3]$

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36. Consider the function defined by $f(x) = \sqrt{x}$; $x \geq 0$. A statement that is true for all real numbers $a, b \geq 0$ is:

(A) $f(a+b) = f(a) + f(b)$
(B) $f(a+b) \neq f(a) + f(b)$
(C) $f(a+b) \leq f(a) + f(b)$
(D) $f(a+b) \geq f(a) + f(b)$
(E) none of these

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37. A square with an area of 20 square units is inscribed in a semicircle. The area of a square that could be inscribed in the entire circle with the same radius is:

(A) 40 sq. units (B) 50 sq. units (C) 60 sq. units (D) 80 sq. units
(E) none of these

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38. The number of real solutions to the equation

$$|2x - 3| = 2 - (2x - 3)^2$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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39. The expression

$$\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}{2}$$

is equal to:

(A) $\cos\left(\frac{\pi}{6}\right)$ (B) $\cos\left(\frac{\pi}{12}\right)$ (C) $\cos\left(\frac{\pi}{24}\right)$ (D) $\cos\left(\frac{\pi}{48}\right)$
(E) $\cos\left(\frac{\pi}{96}\right)$

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40. The number of distinct solutions to the system

$$\begin{aligned} xy + 2xz &= 5\sqrt{5} \\ 2x - (y+2z)w &= 0 \\ 2y - xw &= 0 \\ 2z - 2xw &= 0 \end{aligned}$$

is:

(A) 0 (B) 2 (C) 3 (D) 4 (E) infinite

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41. Let $\tau(n)$ denote the number of positive divisors of the integer n. For example, $\tau(6) = 4$ because the positive divisors of 6 are 1, 2, 3, and 6. The smallest positive integer n for which $\tau(n) = 8$ is:

(A) 2^8 (B) 2^7 (C) 30 (D) 12 (E) none of these

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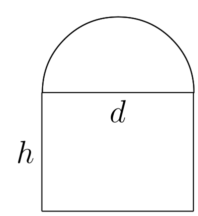
42. A glass window is composed of a rectangle that is h feet high by d feet wide and a semicircle with diameter d, as shown in the figure. If the area of the semicircular region is 1/3 of the total area of the window, then the ratio of h to d is:

(A) $\pi : 2$
(B) $\pi : 3$
(C) $\pi : 4$
(D) $\pi : 5$
(E) none of these



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43. If $x - \frac{1}{x} = 5$, then $x^3 - \frac{1}{x^3}$ is equal to:

(A) 100 (B) 120 (C) 125 (D) 130 (E) none of these

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44. For every natural number $n > 1$, it can be shown that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}.$$

Combining this fact with more elementary properties of logarithms, one can conclude that:

(A) $1/2 < \ln(7/5) < 1$ (B) $0 < \ln(7/5) < 7/6$ (C) $-1 < \ln(7/5) < 0$
(D) $-103/210 < \ln(7/5) < 1/3$ (E) none of these

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45. Venus and Serena are evenly matched at tennis. On any given day, the probability that Venus will beat Serena equals the probability that Serena will beat Venus. There are no ties in tennis. If the two of them play once a day for one week (7 days), then the probability that Venus will win on at least three consecutive days sometime during the week is:

(A) $\frac{9}{64}$ (B) $\frac{47}{128}$ (C) $\frac{3}{8}$ (D) $\frac{5}{8}$ (E) none of these

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46. Let r, s, and t be the (complex-valued) roots of x^3+3x+1. Of the following polynomials, the one whose roots are $r+2$, $s+2$, and $t+2$ is:

(A) $x^3 + 3x + 3$ (B) $x^3 + 3x - 1$ (C) $2x^3 + 6x + 2$
(D) $x^3 - 6x^2 + 15x - 13$ (E) none of these

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47. Circles with radii of 5, 10, and 15 units, respectively, are positioned so that each is externally tangent to the other two. Let A, B, and C be the points of tangency. The perimeter of $\triangle ABC$, in units, is:

(A) 15 (B) 20 (C) $30\sqrt{2}$ (D) $3\sqrt{10}+4\sqrt{5}+5\sqrt{2}$ (E) none of these

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48. The product of the first n natural numbers is denoted by $n!$. The number of consecutive zeros at the end of 250! is:

(A) 25 (B) 27 (C) 50 (D) 62 (E) 64

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49. If the real number r is a root of multiplicity three of the equation $ax^3+bx^2+cx+d=0$ where a, b, c, and d are real numbers with $a \neq 0$, then among the following statements, the one that must be true is:

(A) $r = \dfrac{b}{3a}$

(B) $r^2 = \dfrac{-d}{a}$ and $r^3 = \dfrac{c}{3a}$

(C) $b^2 - 4ac - 3a^2r^2 - 2abr > 0$

(D) $b^2 - 4ac - 3a^2r^2 - 2abr < 0$

(E) $b^2 - 4ac - 3a^2r^2 - 2abr = 0$

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50. Two circles of radius 5 and one circle of radius 8 are positioned so that each is externally tangent to the other two. A fourth circle is circumscribed about the assemblage (i.e., the other three circles are internally tangent to it). The radius of this fourth circle is:

(A) 17 (B) $16\frac{5}{9}$ (C) $13\frac{1}{3}$ (D) $12\frac{1}{2}$ (E) none of these

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Answer Key

1. A
2. D
3. D
4. A
5. D
6. C
7. E
8. C
9. C
10. B
11. B
12. A
13. E
14. C
15. A
16. A
17. C
18. D
19. D
20. D
21. C
22. B
23. A
24. E
25. E
26. C
27. B
28. A
29. A
30. B
31. A
32. E
33. E
34. D
35. B
36. C
37. B
38. C
39. D
40. B
41. E
42. C
43. E
44. B
45. B
46. D
47. D
48. D
49. E
50. C