Mathematics Competition

Indiana University of Pennsylvania

2002

Directions:

1. Please listen to the directions on how to complete the information needed on the answer sheet.

2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.

3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.

4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.

5. Use of pencil, eraser, and scratch paper only are permitted.

6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. If the radius of a circle is increased by 100%, then the area is increased by:

(A) 100% (B) 200% (C) 300% (D) 400% (E) none of these

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2. If the ratio of $5x - 2y$ to $3x + 4y$ is $2/3$, then the ratio of x to y is:

(A) 10/9 (B) 2/3 (C) 2/9 (D) 14/9 (E) none of these

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3. Define a function whose domain is the set of positive integers by

$$F(n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } n = 2 \\ 2 & \text{if } n = 3 \\ \dfrac{F(n-1) \cdot F(n-2) + 2}{F(n-3)} & \text{if } n \geq 4 \end{cases} .$$

Then F(6) is equal to:

(A) 10 (B) 11 (C) 21 (D) 26 (E) none of these

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4. If $2x^{-1} - 1$ is divided by $x - 2$, then the result is:

(A) $\dfrac{1}{x}$ (B) $\dfrac{1}{x-1}$ (C) $\dfrac{-1}{x}$ (D) $\dfrac{1}{1-x}$ (E) none of these

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5. If $90^\circ < \theta < 180^\circ$ and $\cos\theta = -2/3$, then the value of $\tan\theta$ is:

(A) $\dfrac{\sqrt{5}}{2}$ (B) $\dfrac{-\sqrt{5}}{2}$ (C) $\dfrac{2\sqrt{5}}{5}$ (D) $\dfrac{-2\sqrt{5}}{5}$ (E) none of these

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6. The solution to the inequality $6x^2 - 5x < 4$ is:

(A) $-4/3 < x < 1/2$ (B) $-1/2 < x < 4/3$ (C) $-2 < x < 1$
(D) $x < -4/3$ or $x > 1/2$ (E) none of these

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7. If the measure of the vertex angle of an isosceles triangle is 24° less than the sum of the measures of its base angles, then the measure of one of its base angles is:

(A) 156° (B) 66° (C) 24° (D) 78° (E) 51°

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8. The number of real values of x satisfying the equation

$$\frac{3x^2 - 12x}{x^2 - 4x} = x - 1$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) none of these

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9. If $\log_{10} m = b - \log_{10} n$, then the value of m is:

(A) $\dfrac{b}{n}$ (B) bn (C) $b - 10^n$ (D) $10^b n$ (E) $\dfrac{10^b}{n}$

---

10. If $x = 0.12341234\ldots = 0.\overline{1234}$, and $y = 0.123456123456\ldots = 0.\overline{123456}$, then the number of digits in the repeating cycle of the sum $x + y$ is:

(A) 2 (B) 6 (C) 10 (D) 12 (E) 24

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11. The number of terms in the expansion of $[(a + 3b)^3(a - 3b)^3]^3$ when simplified is:

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

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12. The numerical value of $\sin^{-1}(\sin(8\pi/3))$ is:

(A) $\dfrac{8\pi}{3}$ (B) $\dfrac{\pi}{3}$ (C) $\dfrac{-\pi}{3}$ (D) $\dfrac{\sqrt{3}}{2}$ (E) none of these

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13. If $1 - \dfrac{2a}{b + a} = 3$, then $\dfrac{a^{-1} + b^{-1}}{a^{-1} - b^{-1}}$ is equal to:

(A) $1/3$ (B) $-1/3$ (C) 3 (D) -3 (E) none of these

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14. If $f(x + 2) = 3^x + x^3$, then $f(-1)$ is equal to:

(A) $\dfrac{-2}{3}$ (B) -4 (C) $\dfrac{-730}{27}$ (D) $\dfrac{-728}{27}$ (E) -54

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15. The equation $x + \sqrt{x - 2} = 4$ has:

(A) two real roots (B) one real and one imaginary root
(C) two imaginary roots (D) no roots
(E) only one root

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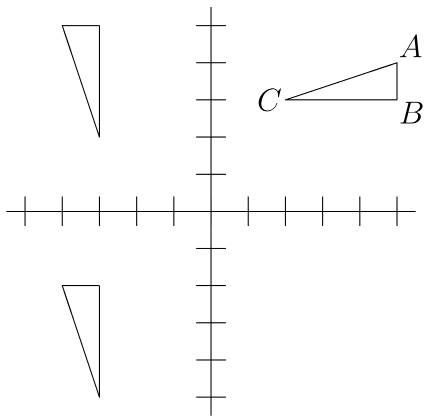
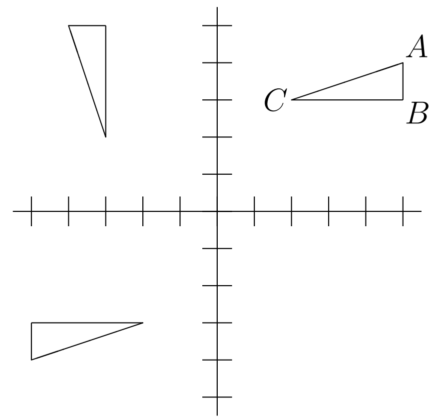
16. A double rotation of $\triangle ABC$ around the origin is represented by:

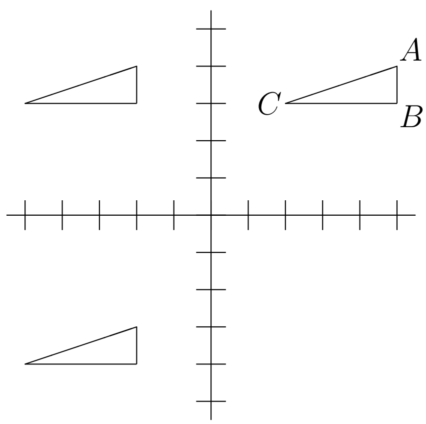
(A)



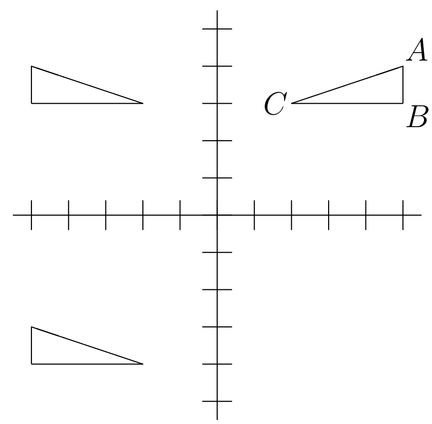
(B)



(C)



(D)



(E) none of these

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17. A male bee has only a mother, but a female bee has both a mother and a father. Assuming all ancestors are distinct, the number of great, great, great grandparents that a female bee has is:

(A) 8 (B) 13 (C) 15 (D) 21 (E) 32

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18. From an original group of boys and girls, twenty girls leave. There are then left two boys for each girl. After this, sixty boys leave. There are then two girls for each boy. The number of girls in the original group was:

(A) 81 (B) 72 (C) 68 (D) 60 (E) 50

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19. A parallelogram has sides of lengths 7 units and 2 units. If its area is $7\sqrt{3}$ square units, then the measure of the smaller angle is:

(A) $30°$ (B) $45°$ (C) $60°$ (D) $75°$ (E) $80°$

---

20. The number of solutions to the equation $e^x + 3e^{-x} = e^{-2x} + 3$ is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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21. If the sum of the areas of the faces of a regular tetrahedron is $24\sqrt{3}$ cm^2, then the volume of that tetrahedron is:

(A) $8\sqrt{3}$ cm^3 (B) $10\sqrt{3}$ cm^3 (C) $24\sqrt{3}$ cm^3 (D) $48\sqrt{3}$ cm^3
(E) $96\sqrt{3}$ cm^3

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22. A man's outfit consists of a pair of pants, a shirt, and a tie. Ron owns five pairs of pants, eight shirts (four solid and four striped), and seven ties (four solid and three striped). The number of possible outfits that Ron can create if he refuses to wear a striped shirt with a striped tie is:

(A) 60 (B) 80 (C) 140 (D) 160 (E) none of these

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23. The solution set, in interval notation, for the inequality $||x-2|-3| < 7$ is:

(A) $[0, 10)$ (B) $(-8, 12)$ (C) $(-\infty, 0]$ (D) $(-2, 6)$ (E) $[0, 12]$

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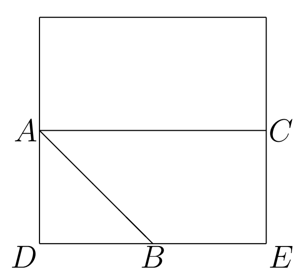
24. In the figure, A, B, and C are midpoints of the sides of the square. If the area of $\triangle ADB$ is 5, then the perimeter of the polygon $ABEC$ is:

(A) $4\sqrt{10} + 2\sqrt{5}$
(B) $8\sqrt{5}$
(C) $20 + 4\sqrt{5}$
(D) $40 + 4\sqrt{5}$
(E) 25



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25. If m men can do a job in d days, then the number of days it takes $m + r$ men to do the job is:

(A) $d + r$ (B) $d - r$ (C) $\dfrac{d}{m+r}$ (D) $\dfrac{md}{m+r}$ (E) none of these

26. The figure shows the graph of $f(x) = \ln x$. Suppose $P = (1, 0)$ and $Q = (x, 0)$ where $x > 1$. If the area of the rectangle $PQRS$ is $5 - \ln x$, then the value of x^x is:

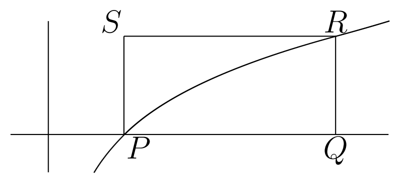
(A) e

(B) $\ln 5$

(C) e^2

(D) $\ln e^3$

(E) e^5



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27. If $\sin^{-1} x + \sin^{-1} y = \pi/2$, then the numerical value of $x^2 + y^2$ is:

(A) 0 (B) 1 (C) $\pi/2$ (D) π (E) none of these

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28. The quadratic expression $21x^2 + ax + 21$ can be factored into a product of two binomial linear factors with integer coefficients if a is:

(A) any odd integer (B) some odd integer (C) any even integer
(D) some even integer (E) zero

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29. The number of integral points that lie on or inside a circle centered at (0,0) with a radius of 3 is:

(A) 21 (B) 25 (C) 28 (D) 29 (E) none of these

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30. The number of solutions to the equation $x^2 + |x| - 2 = 0$ is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

---

31. For any nonnegative integer n, we define $n!$ by

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n(n-1)(n-2)\cdots 2 \cdot 1 & \text{if } n \geq 1 \end{cases}.$$

Also we define the symbol

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}.$$

If $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{x}$, then x is equal to:

(A) k (B) n (C) 1 (D) $k+1$ (E) none of these

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32. The product of all real roots of the equation $x^{\log_{10} x} = 10$ is:

(A) 1 (B) -1 (C) 10 (D) 10^{-1} (E) none of these

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33. The solution of the inequality

$$\frac{x^2(x-3)}{x^2+4x+4} \le 0$$

is:

(A) $x < -2$ or $-2 < x < 0$ or $0 < x < 3$ (B) $x \ge 3$
(C) $x < 3$ and $x \ne -2$ (D) $-2 < x \le 0$ or $x \ge 3$
(E) $x < -2$ or $-2 < x \le 3$

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34. The set of all solutions of the system

$$\begin{aligned} x+y &\le 4 \\ x+y &\ge 2 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

is a trapezoidal region. The perimeter of this region is:

(A) $10\sqrt{2}$ (B) 20 (C) $8+4\sqrt{2}$ (D) $12\sqrt{2}$ (E) $4+6\sqrt{2}$

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35. A merchant buys goods at 25% off the list price. She desires to mark the goods so she can give a discount of 20% on the marked price and still clear a profit of 25% of the list price. As a percentage of the list price, the marked price is:

(A) 80% (B) 90% (C) 100% (D) 120% (E) 125%

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36. If $f(3x+7) = \dfrac{5}{4x-3}$, then $f(x)$ is equal to:

(A) $\dfrac{15}{4x-37}$ (B) $\dfrac{5}{12x+25}$ (C) $\dfrac{5}{12x-31}$ (D) $\dfrac{5}{3(4x-3)} - 7$

(E) none of these

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37. Suppose the word WATCH is spelled out on scrabble tiles. If a person randomly arranges the tiles, then the probability that the word CAT will be formed within the sequence of five tiles is:

(A) $\dfrac{1}{120}$ (B) $\dfrac{1}{40}$ (C) $\dfrac{1}{20}$ (D) $\dfrac{1}{12}$ (E) none of these

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38. The number of real solutions to the equation

$$\left(\sqrt{x-1}-x+7\right)\left(\sqrt{2x+3}-\sqrt{x+2}-2\right)=0$$

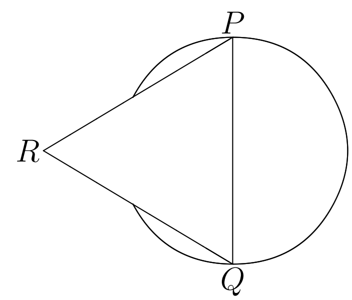
is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

---

39. In the figure, $\triangle PQR$ is an equilateral triangle and PQ is a diameter of the circle. If the length of this diameter is 8, then the area of the shaded region is:

(A) $8\pi - 12\sqrt{3}$
(B) $8\pi - 16$
(C) $\dfrac{16\pi}{3} - 8\sqrt{3}$
(D) $\dfrac{8\pi}{3} - 4\sqrt{3}$
(E) none of these



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40. The solution, in interval notation, to the inequality

$$\frac{\sqrt{x+2}}{x-5} \geq 0$$

is:

(A) $(5,\infty)$ (B) $[-2,5]$ (C) $[-2,5)$ (D) $(-\infty,-2]\cup(5,\infty)$
(E) none of these

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41. A steel beam 500 ft in length lies on a level surface and is securely fastened at both ends. Heat causes the beam to expand in the form of a circular arc. If the highest point on the beam is one foot above ground level, then the length in feet of the expanded beam is:

(A) $62501\sin^{-1}\left(\dfrac{500}{62501}\right)$ (B) 501 (C) $\dfrac{62501}{2}\sin^{-1}\left(\dfrac{500}{62501}\right)$

(D) $500\sin^{-1}\left(\dfrac{500}{62501}\right)$ (E) none of these

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42. If $2\log_4(x-2y)=\log_4 x+\log_4 y$, then the set of possible values for x/y is:

(A) $\{0,4\}$ (B) $\{1,4\}$ (C) $\{-1,-4\}$ (D) $\{1,5\}$ (E) none of these

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43. A woman walked a certain distance at a constant rate. If she had gone 1/2 mile per hour faster, then she would have walked the same distance in four-fifths of the time. On the other hand, if she had gone 1/2 mile per hour slower, then she would have been two hours longer on the road. The distance in miles that she walked was:

(A) 10 (B) 12 (C) 13.5 (D) 15 (E) none of these

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44. The percentage of all natural numbers that are divisible by 2, 3, or 5 is closest to:

(A) 70% (B) 71% (C) 72% (D) 73% (E) 74%

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45. The graph of the equation

$$x(x^2 - xy + y^2 - 1) = y(y^2 - 1)$$

consists of:

(A) one circle and one line
(B) one parabola and one line
(C) two lines and one ellipse
(D) three lines
(E) two parabolas

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46. The number of real solutions to the equation

$$|x^2 - 4x + 3| + 1 = |x - 1| + |x - 3|$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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47. Ten cards, numbered one through ten, are shuffled well, and three cards are randomly chosen from the ten. The probability that the smallest number on a chosen card is at most three and the largest number on a chosen card is at least 8 is:

(A) $\frac{3}{10}$ (B) $\frac{9}{20}$ (C) $\frac{289}{576}$ (D) $\frac{7}{12}$ (E) $\frac{3}{5}$

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48. If $x = a(\sec\theta + \tan\theta)$, then the expression $\dfrac{2ax}{x^2 + a^2}$ is equivalent to:

(A) $\sin\theta$ (B) $\cos\theta$ (C) $\tan\theta$ (D) $\sec\theta$ (E) none of these

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49. The sum of the coefficients of the terms involving all of the variables x, y, and z, each raised to a positive power in the expansion of $(x + y + z)^5$ is:

(A) 94 (B) 120 (C) 150 (D) 180 (E) 243

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50. A circle is inscribed in a 30°-60°-90° triangle whose shortest leg is one unit long. The area of this circle, in square units, is:

(A) $\left(\sqrt{3}-1\right)\pi$

(B) $\left(\dfrac{2-\sqrt{3}}{2}\right)\pi$

(C) $\left(\dfrac{6-3\sqrt{2}-2\sqrt{3}+\sqrt{6}}{12}\right)\pi$

(D) $\left(\dfrac{6-4\sqrt{2}-3\sqrt{3}+2\sqrt{6}}{48}\right)\pi$

(E) none of these

Answer Key

1. C
2. D
3. C
4. C
5. B
6. B
7. E
8. A
9. E
10. D
11. D
12. B
13. A
14. D
15. E
16. B
17. B
18. D
19. C
20. B
21. A
22. E
23. B
24. A
25. D
26. E
27. B
28. D
29. D
30. C
31. A
32. A
33. E
34. E
35. E
36. A
37. C
38. C
39. C
40. E
41. A
42. E
43. B
44. D
45. A
46. D
47. B
48. B
49. C
50. B