Mathematics Competition

Indiana University of Pennsylvania

2003

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.

2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.

3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.

4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.

5. Use of pencil, eraser, and scratch paper only are permitted.

6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. If the height of a right circular cylinder is tripled and the diameter is halved, then the ratio of the new volume to the original volume is:

   (A) 4/3 (B) 3/4 (C) 3/2 (D) 2/3 (E) none of these

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2. An expression that is equivalent to $3 - \dfrac{3-a}{3-a/3}$ is:

   (A) $\dfrac{18}{9-a}$ (B) $\dfrac{18-6a}{9-a}$ (C) $a - a/3$ (D) $18 - 9/a$
   (E) none of these

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3. The complex number $(1+i)^4(1-i)^3$ is equal to:

   (A) $1+i$ (B) $1-i$ (C) $2+2i$ (D) $4-4i$ (E) $8+8i$

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4. The expression $\sqrt{18x} - \sqrt{8x}$ is equivalent to:

   (A) $\sqrt{10x}$ (B) $2\sqrt{5x}$ (C) $2x\sqrt{5}$ (D) $\sqrt{2x}$ (E) none of these

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5. The numerical value of $\sin\left(\cot^{-1}\frac{4}{5}\right)$ is:

   (A) $\dfrac{4}{\sqrt{41}}$ (B) $\dfrac{5}{\sqrt{41}}$ (C) $\dfrac{3}{5}$ (D) $\dfrac{4}{5}$ (E) none of these

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6. If the ratio of $y + 4x$ to $5x - y$ is 2/3, then the ratio of x to y is:

   (A) $-5/2$ (B) $-1/2$ (C) 2/5 (D) 5/6 (E) none of these

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7. An equation for a line perpendicular to the graph of $3y - 4x = 5$ is:

   (A) $3y + 4x = 7$ (B) $4y - 3x = 6$ (C) $3y = 4x + 5$ (D) $4y = -3x + 4$
   (E) none of these

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8. It took George and Martha, working together, 6 days to paint their house. If Martha could have painted it in 10 days by herself, the number of days it would have taken George working alone is:

   (A) 4 (B) 7 (C) 12.5 (D) 14 (E) none of these

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9. The sum of the geometric series $1 - 1/2 + 1/4 - 1/8 + \cdots$ is:

(A) 0 (B) 1/2 (C) 2/3 (D) 1 (E) 2

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10. The numerical value of $(\log_2 3)(\log_3 4)$ is:

(A) 2/3 (B) 3/4 (C) 1 (D) 2 (E) $\log_6 12$

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11. When three numbers are added two at a time, the sums are 33, 48, and 53. The sum of all three numbers is:

(A) 134 (B) 86 (C) 81 (D) 67 (E) 53

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12. On a clock, the measure of the angle between the hour and minute hands at 5:15 a.m. is:

(A) $60°$ (B) $67.5°$ (C) $75°$ (D) $82.5°$ (E) none of these

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13. If the equation $-2x^3 + ax^2 + bx + c = 0$ has roots -2, 3, and 6, then the values of a, b, and c are:

(A) $a = -7$, $b = 0$, $c = 36$
(B) $a = 7$, $b = 0$, $c = -36$
(C) $a = -14$, $b = 0$, $c = 72$
(D) $a = -14$, $b = 0$, $c = -72$
(E) $a = 14$, $b = 0$, $c = -72$

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14. Let $f(x) = 2x^3 - 7$. If the function $g(x)$ satisfies $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, then $g(x)$ is:

(A) $\dfrac{1}{2x^3 - 7}$ (B) $\dfrac{7}{2\sqrt[3]{x}}$ (C) $\dfrac{\sqrt[3]{x} + 7}{2}$ (D) $\sqrt[3]{\dfrac{x+7}{2}}$

(E) not uniquely determined

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15. John starts a new job. He works 8 hours the first day. On the second day he works the same time but gets twice as much done. On the third day he again works the same time but finishes twice as much as he did the second day. The number of hours it would have taken John to do the work he accomplished in three days if he worked at his initial rate is:

(A) 48 (B) 50 (C) 56 (D) 60 (E) 72

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16. The numerical value of $\sin 105^\circ$ is:

(A) $\dfrac{\sqrt{2}-\sqrt{6}}{4}$ (B) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ (C) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ (D) $\dfrac{\sqrt{2-\sqrt{2}}}{4}$

(E) none of these

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17. A rectangle is inscribed in a circle of radius 6.5 cm. If the the length of the rectangle is 2 cm longer than twice the width, then the dimensions of the rectangle are:

(A) 4.5 cm × 11 cm (B) 5 cm × 12 cm (C) 5.5 cm × 13 cm
(D) 6.5 cm × 12 cm (E) none of these

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18. The sum of two numbers is a and the product of the same two numbers is b. The sum of the cubes of these numbers is:

(A) $a^3 - 3ab$ (B) $a^3 + 3ab$ (C) $b^3 - 3ab$ (D) $a^3 - b^3$
(E) none of these

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19. To number the pages of a volume, 3001 digits were printed. Assuming all pages are numbered, the number of pages is:

(A) 1021 (B) 1026 (C) 1027 (D) 1028 (E) none of these

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20. A number equal to $\dfrac{1}{\sqrt[3]{4}+1}$ is:

(A) $\dfrac{\sqrt[3]{4}-1}{3}$ (B) $\dfrac{\sqrt[3]{4}-1}{5}$ (C) $\dfrac{\sqrt[3]{16}-\sqrt[3]{4}+1}{3}$ (D) $\dfrac{\sqrt[3]{16}+\sqrt[3]{4}+1}{5}$

(E) $\dfrac{\sqrt[3]{16}-\sqrt[3]{4}+1}{5}$

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21. If $x = \log_2 6$ and $y = 3^{12}$, then the numerical value of

$$\left(\frac{2^{3x} - \log_3 y}{12}\right)^2$$

is:

(A) 289 (B) 204 (C) 144 (D) 36 (E) none of these

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22. The number of positive integer divisors of 1,000,000 is:

(A) 1,000,000 (B) 1000 (C) 49 (D) 36 (E) 25

---

23. The number of values of a such that $x^2 - 9x + a$ has two distinct positive integer roots is:

(A) 0 (B) 4 (C) 5 (D) 9 (E) infinite

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24. The minimum number of obtuse angles in a convex pentagon is:

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

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25. The solution set of the equation $|2 - |x - 7|| = 3$ is:

(A) $\{6\}$ (B) $\{2\}$ (C) $\{2, 12\}$ (D) $\{2, 6, 12\}$ (E) none of these

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26. Let $x = 2\sin\theta$ where $\pi/2 < \theta < \pi$. The expression $\sqrt{4 - x^2}$ is equal to:

(A) $2 - 2\sin\theta$ (B) $2\cos\theta$ (C) $-2\cos\theta$ (D) $-2\sin\theta$
(E) none of these

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27. The area of a regular hexagon with a perimeter of 10 cm is:

(A) $\frac{25\sqrt{3}}{6}$ cm^2 (B) $5\sqrt{3}$ cm^2 (C) $\frac{5\sqrt{3}}{6}$ cm^2 (D) $\frac{5\sqrt{3}}{12}$ cm^2

(E) $\frac{3\sqrt{3}}{2}$ cm^2

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28. The sum of the solutions of the equation $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$ is:

(A) -2 (B) 0 (C) 2 (D) 8 (E) 10

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29. If $f(x) = \frac{3}{x-1}$ and $f(g(x)) = \frac{x}{5x-3}$, then $g(x)$ is:

(A) $16 - \frac{9}{x}$ (B) $\frac{3x}{5x-3}$ (C) $\frac{14x-9}{x}$ (D) $\frac{x(x-1)}{3(5x-3)}$

(E) none of these

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30. The solution of the inequality $\dfrac{3^{2x}-1}{3-x} \leq 0$ is:

(A) $x < 0$ (B) $x \leq 0$ or $x > 3$ (C) $0 \leq x < 3$ (D) $x > 3$
(E) $0 < x < 3$

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31. The number of real solutions of the equation

$$x(x^2+1)(1+\ln x)(e^{-x}-2) = 0$$

that exceed 1 in absolute value is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) none of these

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32. There are 55 seats in the front row of the local theater. When Martha enters she notices that she cannot sit in the front row without sitting next to someone who is already there. The fewest number of people who could possibly already be seated is:

(A) 18 (B) 19 (C) 27 (D) 28 (E) none of these

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33. The solution set of the equation $\sqrt{5x+6} = 3 + \sqrt{x+3}$ is:

(A) $\{-3/4, 6\}$ (B) $\{6\}$ (C) $\{-3/4\}$ (D) $\emptyset$ (E) none of these

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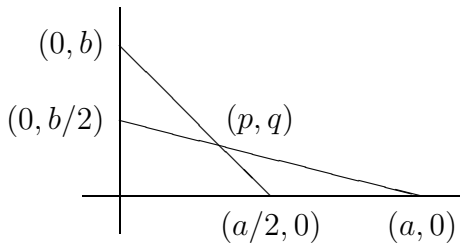
34. If (p, q) is the point of intersection of the lines in the accompanying figure, then $2p+q$ is equal to:

(A) $\dfrac{2a+b}{3}$
(B) $\dfrac{a+2b}{3}$
(C) $\dfrac{2a+b}{2}$
(D) $2a+b$
(E) none of these



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35. A man walks down a hill, along a level road, and then up another hill. At the top of the second hill, he turns around and returns to where he started. His entire walk took him 4 hours. If he walks 6 mph downhill, 3 mph on level ground, and 2 mph uphill, then the total distance he walked is:

(A) 6 miles (B) 12 miles (C) 16 miles (D) impossible to determine
(E) none of these

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36. The radian measure of the acute angle determined by the lines $y = 2x$ and $y = -3x$ is:

(A) $\frac{\pi}{7}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{5}$ (D) $\frac{\pi}{4}$ (E) none of these

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37. The solution set to the equation $\log_2(3-x) + \log_2(1-x) = 3$ is:

(A) $\{1/2\}$ (B) $\{-2\}$ (C) $\{-1, 5\}$ (D) $\{5\}$ (E) none of these

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38. The sum of two numbers is 10. The smallest possible value of the sum of the square of the first number plus two times the second number is:

(A) 9 (B) 18 (C) 19 (D) 20 (E) 23

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39. Consider the line segment from $(1, 0)$ to $(1, 1)$ cut into two parts by a ray from the origin that makes an angle of 30 degrees with the x-axis. The length of the smaller portion divided by the length of the larger portion is:

(A) $\sqrt{3}+1$ (B) $\frac{\sqrt{3}+1}{\sqrt{3}}$ (C) $\frac{\sqrt{3}}{\sqrt{3}+1}$ (D) $\sqrt{3}-1$ (E) none of these

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40. The solution set of the equation $|x+2| + |x-1| > 1$ is:

(A) $\{x \mid -1 < x < 2\}$ (B) $\{x \mid x < -2 \text{ or } x > 1\}$ (C) $\{x \mid -2 < x < 1\}$
(D) $\{x \mid x > -2\}$ (E) none of these

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41. For a positive integer n, define $n!$ to be the product of the first n positive integers. For example, $1! = 1$, $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$, etc. The number of zero digits at the end of 25! is:

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

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42. Suppose that the sides of a triangle are in the ratio $4 : 5 : 6$. The sine of the smallest angle is:

(A) $\frac{2}{3}$ (B) $\frac{3}{4}$ (C) $\frac{5}{6}$ (D) $\frac{\sqrt{15}}{4}$ (E) $\frac{\sqrt{7}}{4}$

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43. The solution set of the inequality $x - 2x^{2/3} - x^{1/3} + 2 < 0$ is:

(A) $(-1, 1) \cup (2, \infty)$ (B) $(-\infty, -1) \cup (8, \infty)$ (C) $(-\infty, -1) \cup (1, 2)$
(D) $(-\infty, -1) \cup (1, 8)$ (E) none of these

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44. For the experiment of rolling three ordinary six-sided dice and recording the sum, the probability of obtaining a sum between 9 and 12 inclusive is:

(A) $\frac{1}{4}$ (B) $\frac{7}{27}$ (C) $\frac{13}{27}$ (D) $\frac{14}{27}$ (E) $\frac{16}{27}$

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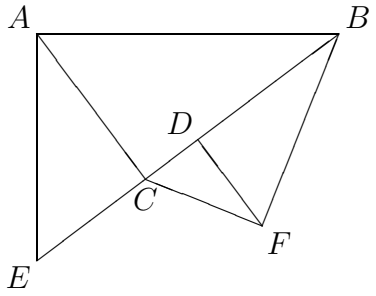
45. In the figure below, $\angle BAE$, $\angle ACB$, $\angle BDF$, and $\angle CFB$ are right angles. If $\overline{DB} = 6$, $\overline{AB} = 10$, and $\overline{DF} = 2\sqrt{3}$, then the sum $\overline{CD} + \overline{EC} + \overline{AC} + \overline{AE}$ is:

(A) 20
(B) 24
(C) 36
(D) $\frac{16\sqrt{3} + 24}{3}$
(E) $\frac{24\sqrt{3} + 14}{3}$



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46. The number of solutions of the system

$$\begin{aligned} 3x^2 + 2xy &= 3 \\ 3x^2 + 2y^2 &= 5 \end{aligned}$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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47. Suppose that five candy bars are randomly distributed among three children so that each child is equally likely to receive each candy bar. The probability that each child will receive at least one candy bar is:

(A) $\frac{2}{7}$ (B) $\frac{49}{81}$ (C) $\frac{50}{81}$ (D) $\frac{80}{81}$ (E) $\left(\frac{211}{243}\right)^3$

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48. The solution to the equation $\log_2 x + \log_3 x = 5$ is:

(A) $32^{\frac{\ln 3}{\ln 2+\ln 3}}$ (B) $32^{\frac{\ln 2}{\ln 2+\ln 3}}$ (C) $32^{\frac{\ln 2+\ln 3}{\ln 3}}$ (D) $32^{\frac{\ln 2+\ln 3}{\ln 2}}$
(E) none of these

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49. If $\frac{x+1}{x^3-5x^2+8x-4} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$, then $A+2B+3C$ is equal to:

(A) -4 (B) 0 (C) 2 (D) 7 (E) 10

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50. A polygon is formed by the lines $4y = 3x$, $18x + y = -75$, and $6x + 17y = 225$. The area interior to this polygon and exterior to the curve $x^2 + y^2 - 8x - 6y = 0$ is:

(A) $300 - 25\pi$ (B) $150 - 25\pi$ (C) $150 + \frac{25}{2}\pi$ (D) $150 - \frac{25}{2}\pi$
(E) none of these

Answer Key

1. B
2. A
3. E
4. D
5. B
6. A
7. D
8. E
9. C
10. D
11. D
12. B
13. E
14. D
15. C
16. C
17. B
18. A
19. C
20. E
21. A
22. C
23. B
24. B
25. C
26. C
27. A
28. E
29. A
30. B
31. A
32. B
33. B
34. A
35. B
36. D
37. E
38. C
39. D
40. E
41. E
42. E
43. D
44. C
45. A
46. E
47. C
48. A
49. D
50. D