Mathematics Competition
Indiana University of Pennsylvania
2004

Directions:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. The number of positive integers n such that $n^2 - 3n + 2$ is prime is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) infinite

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2. The value of k for which $2^{2004} - 2^{2003} + 2^{2002} + 2^{2001} = k2^{2000}$ is:

(A) 2 (B) 14 (C) 18 (D) 29 (E) 30

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3. The area of the rectangle formed by the lines $y = x$, $y = x+6$, $y = -x$, and $y = -x+2$ is:

(A) 6 (B) 2 (C) 12 (D) 3 (E) none of these

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4. Suppose x is 40% more than y and y is 20% more than z. The percentage by which x exceeds z is:

(A) 48 (B) 50 (C) 60 (D) 68 (E) 88

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5. The value of x that satisfies $x \log_2 4 = 26$ is:

(A) 13 (B) 9 (C) 13/2 (D) 2 (E) none of these

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6. The solution, in interval notation, to the statement

$$|2 - 3x| > 5 \quad \text{or} \quad 3x - 2 = 5$$

is:

(A) $(-\infty, -1) \cup [7/3, \infty)$ (B) $(-1, 7/3]$ (C) $(-\infty, 7/3]$ (D) $(-1, \infty)$
(E) $(-\infty, \infty)$

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7. Two fair six-sided dice are to be rolled. The probability of obtaining a difference that is less than or equal to two in absolute value is:

(A) 1/6 (B) 1/3 (C) 1/2 (D) 2/3 (E) 3/4

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8. If $\dfrac{2x + 3y}{4x - 5y} = 8$, then y/x equals:

(A) 1 (B) $-5/2$ (C) 3/4 (D) 4/15 (E) 30/43

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9. The distance from the point $(4, 0)$ to the line $y = x + 2$ is:

(A) 3 (B) 4 (C) $3\sqrt{2}$ (D) $2\sqrt{5}$ (E) 5

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10. The graph of $y = 2\tan(18x)$ has a period of:

(A) 18 (B) 9 (C) $\pi/18$ (D) 36 (E) π

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11. Let $f(x) = (x+1)^x(x+3)^{x+4}$. The value of $f(0) + f(-4)$ is:

(A) $81\frac{1}{81}$ (B) $80\frac{80}{81}$ (C) 1 (D) 2 (E) 80

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12. Let $F_1, F_2, F_3, \ldots$ be a sequence of numbers that satisfies the equations $F_1 = F_2$ and $F_n = F_{n-2} + F_{n-1}$ for $n \geq 3$. For positive integers $n \geq 3$, define

$$g(n) = F_1 - F_2 + F_3 - F_4 + \cdots + (-1)^{n+1}F_n.$$

If n is an odd integer with $n \geq 3$, then the value of $g(n) - g(n+1)$ is:

(A) F_n (B) $-F_n$ (C) F_{n+1} (D) $-F_{n+1}$ (E) 0

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13. The solution, in interval notation, to the inequality

$$\frac{x^2+1}{x^2(x-3)} \leq 0$$

is:

(A) $(3, \infty)$ (B) $(-\infty, 0) \cup (0, 3]$ (C) $(0, 3)$ (D) $(-\infty, 3)$
(E) $(-\infty, 0) \cup (0, 3)$

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14. The value of $\sqrt{\log_{10}(\text{googol})}$ is:

(A) 2 (B) 10 (C) 100 (D) 1000 (E) none of these

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15. John leaves for work at 8:30 a.m. every day. If he drives 30 mph, he is 5 minutes later than his target arrival time. If he drives 50 mph, he is 5 minutes earlier than this target arrival time. The speed at which he should drive to hit his target arrival time exactly is:

(A) 35 mph (B) 37.5 mph (C) 40 mph (D) 42.5 mph (E) 45 mph

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16. An ant is crawling on the surface of a unit cube. The shortest distance for the ant to crawl from one corner to the opposite corner of the cube is:

(A) $\sqrt{3}$ (B) $\sqrt{5}$ (C) $\sqrt{10}$ (D) $2\sqrt{3}$ (E) $\sqrt{17}$

---

17. If $a = \ln N$, $b = \ln N^2$, and $c = \ln N^4$ where $N > 0$, then $\dfrac{ab}{a+b}$ is equal to:

(A) c (B) $c/2$ (C) $c/3$ (D) $c/6$ (E) $c/8$

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18. If $\sqrt{\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right)\cdots\left(\frac{a}{b}\right)} = 20\sqrt{5}$, then $a + b$ is equal to:

(A) 5999 (B) 6999 (C) 7999 (D) 8999 (E) none of these

---

19. The distance between the two x-intercepts of the parabola with the equation

$$y = 5x^2 + 8x - 4$$

is:

(A) 2.4 (B) 2 (C) 1.6 (D) 0.4 (E) 0.2

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20. Including multiplicities, the sum of the roots of the equation

$$x^5 - 6x^4 + 2x^3 + 48x^2 - 99x + 54 = 0$$

is:

(A) 0 (B) 1 (C) 2 (D) 4 (E) 6

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21. The coefficient of x^4y^2 in the expansion of $(2x + 3y)^6$ is:

(A) 15 (B) 90 (C) 2160 (D) 4860 (E) 36

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22. The value of $\cos^2\left[\frac{1}{2}\text{Arcsin}(x)\right]$ is:

(A) $\dfrac{1-\sqrt{1-x^2}}{2}$ (B) $2\sqrt{1-x^2}-1$ (C) $2-\sqrt{1-x^2}$ (D) $\dfrac{\sqrt{1-x^2}-1}{2}$

(E) none of these

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23. A number that is equal to $\dfrac{5}{\sqrt[3]{49}+\sqrt[3]{14}+\sqrt[3]{4}}$ is:

(A) $\sqrt[3]{7}-\sqrt[3]{2}$ (B) $\dfrac{5}{\sqrt[3]{67}}$ (C) $\sqrt[3]{49}-\sqrt[3]{14}+\sqrt[3]{4}$ (D) $\sqrt[3]{7}+\sqrt[3]{2}$ (E) 6

---

24. For the appropriate domains, a function g that satisfies

$$f(x) = \frac{1}{x+2} \quad \text{and} \quad f \circ g(x) = \frac{x-1}{2x+2}$$

is:

(A) $\dfrac{4}{x-1}$ (B) $\dfrac{x+3}{x-1}$ (C) $\dfrac{4x+8}{-x-1}$ (D) $\dfrac{x+2}{2x+5}$ (E) $\dfrac{4x-4}{x-5}$

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25. Let $f(x) = 1 + x + 2x^2 + 3x^3 + \cdots + 20x^{20}$. The coefficient of x^{10} in $(f(x))^2$ is:

(A) 210 (B) 185 (C) 180 (D) 100 (E) none of these

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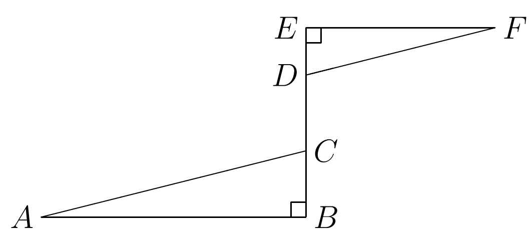
26. In the figure below, $|BE| = |EF| = 5$, $|CD| = 2$, and AC is parallel to DF. If the distance from A to F is 13, then the length of AB is:

(A) 7
(B) 10
(C) 12
(D) $8\sqrt{2}$
(E) none of these



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27. If (x, y) lies on the line with slope 1 and y-intercept $(0, -\ln 2)$, then e^{2x}/e^{2y} is equal to:

(A) 1 (B) 2 (C) 4 (D) 8 (E) 16

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28. The number of distinct solutions to the equation $|2x - |x+1|| = 2$ is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

---

29. The length of the major axis for the ellipse with equation $9x^2 - 36x + y^2 + 6y = 36$ is:

(A) 9 (B) 18 (C) 6 (D) 36 (E) 12

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30. The number of values of a such that $x^2 - 10x + a$ has two distinct integer roots is:

(A) 0 (B) 4 (C) 5 (D) 10 (E) infinite

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31. Consider the set of all strings of 0's and 1's having length 20. An example would be 11110000111100001100. The number of such strings having an even number of 0's is.

(A) 10^2 (B) 10^{19} (C) 2^{19} (D) 2^{10} (E) none of these

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32. Consider four different circles in a plane. The largest number of intersection points possible is:

(A) 14 (B) 16 (C) 8 (D) 10 (E) 12

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33. If $N > 0$, then $\sqrt[5]{N\sqrt[4]{N\sqrt[3]{N}}}$ is equivalent to:

(A) $N^{47/60}$ (B) $N^{7/60}$ (C) $N^{1/60}$ (D) $N^{16/15}$ (E) $N^{4/15}$

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34. The number of positive integers less than or equal to 1501 that are multiples of 3 or 4 but not a multiple of 5 is:

(A) 875 (B) 850 (C) 750 (D) 725 (E) none of these

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35. The sum of the distinct solutions of the equation $x^4 - x^3 - 7x^2 + 13x - 6 = 0$ is:

(A) -6 (B) -1 (C) 0 (D) 1 (E) 6

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36. The number of x-intercepts that the graph of $f(x) = \tan(\log_{10}(x))$ has over the interval $(0, 1)$ is:

(A) 0 (B) 1 (C) 2 (D) 5 (E) infinite

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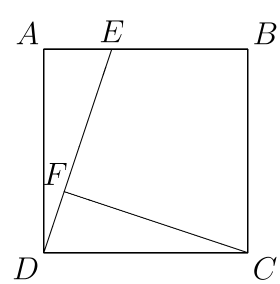
37. Consider the square $ABCD$ with side length 3. If $|AE| = \frac{1}{3}|AB|$ and $\overline{CF} \perp \overline{ED}$, then the area of $EBCF$ is:



(A) 123/20
(B) 15/2
(C) 27/10
(D) 24/5
(E) 9/2

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38. The number of points of intersection of the graphs of the functions

$$f(x) = \begin{cases} 1 - |x-1| & \text{if } 0 \le x \le 2 \\ |x-3| - 1 & \text{if } 2 < x \le 4 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} |x-1| - 1 & \text{if } 0 \le x \le 2 \\ 1 - |x-3| & \text{if } 2 < x \le 4 \end{cases}$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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39. Let x, y, and z be positive integers for which $x \log_{100} 5 + y \log_{100} 2 = z$. A possible value for the sum of x, y, and z is:

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

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40. A bicyclist travels down a hill, along a level road, and then up another hill. At the top of the second hill, she turns around and returns to where she started. Her entire trip took her 6 hours. If she goes 20 mph downhill, 10 mph on level ground, and 5 mph uphill, then the total distance she traveled is:

(A) 50 miles (B) 55 miles (C) 70 miles (D) impossible to determine
(E) none of these

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41. Suppose that a function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the property $f(ax+b) = af(x) + f(b)$ for all $a, b, x \in \mathbb{R}$. The slope of the line passing through any two points on the graph of f is:

(A) 0 (B) 1 (C) $f(0)$ (D) $f(1)$ (E) none of these

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42. If $\tan\alpha = 1/3$ with $\pi < \alpha < 3\pi/2$ and $\cos\beta = 1/3$ with $3\pi/2 < \beta < 2\pi$, then $\sin(\alpha - \beta)$ is equal to:

(A) $\dfrac{\sqrt{10}}{30} - \dfrac{2\sqrt{5}}{5}$ (B) $\dfrac{-\sqrt{10}}{30} - \dfrac{2\sqrt{5}}{5}$ (C) $\dfrac{-\sqrt{10}}{10} + \dfrac{2\sqrt{5}}{15}$

(D) $\dfrac{\sqrt{10}}{10} + \dfrac{2\sqrt{5}}{15}$ (E) $\dfrac{3}{10} - \sqrt{8}$

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43. The number of real-valued 5-tuple solutions (u, w, x, y, z) to the system

$$\begin{aligned} x + y + z &= 10 \\ x^2 - y^2 &= z \\ ux + 2wy &= 0 \\ uy + 2wx &= 0 \\ w - u &= 1 \end{aligned}$$

is:

(A) 0 (B) 1 (C) 4 (D) 5 (E) infinite

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44. Define a *division* of a square to be any curve in the interior of the square whose endpoints are two distinct points on the boundary of the square. Add divisions to a square subject to the following two conditions.

I. A division intersects all previous divisions exactly once.

II. No three divisions contain a common point.

For $n = 0, 1, 2, \ldots$, let R_n denote the numbr of regions in the square after drawing n divisions. The value of R_{20} is:

(A) 2^{19} (B) 2^{20} (C) 210 (D) 211 (E) none of these

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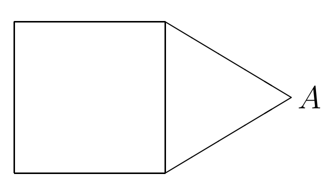
45. A pentagonal storage shed is formed by extending a square shed with an addition in the shape of an equilateral triangle as shown in the figure. A goat is tethered outside the shed to the corner labeled A. Each side of the shed is 6 feet in length and the goat's rope is 18 feet long. Assuming the goat does not chew through the rope, the square footage of the region over which the goat can graze is:

(A) $324\pi - 36 - 9\sqrt{3}$
(B) 312π
(C) $312\pi - 9\sqrt{3}$
(D) $300\pi + 9\sqrt{3}$
(E) none of these



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46. Of the following systems, the one with the greatest number of ordered pair solutions (x, y) with $x, y \in \mathbb{R}$ is:

(A) $\begin{aligned} x+y &= 1 \\ xy &= 1 \end{aligned}$ (B) $\begin{aligned} x-y &= 1 \\ xy &= 1 \end{aligned}$ (C) $\begin{aligned} x+y &= 2 \\ xy &= 1 \end{aligned}$

(D) $\begin{aligned} x+y &= 2 \\ x^2y &= 1 \end{aligned}$ (E) $\begin{aligned} x+y &= 2 \\ x^3y &= 1 \end{aligned}$

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47. If $x > 0$ satisfies $\log_{2004} x + \log_x 2004 = 8$, then $\log_{2004} x$ is equal to:

(A) $4 \ln 2004 \pm \ln 2004$ (B) 4 ± 1 (C) $4 \pm \sqrt{15}$ (D) $501/2$
(E) none of these

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48. Given the functions $R(x) = 4x - 2$, $S(x) = 2x^2$, $T(x) = 6x + 3$, and $U(x) = x^2$, the composition of all four functions (each used just once) that will produce the largest value when evaluated at $x = 3$ is:

(A) $U(T(S(R(x))))$ (B) $S(U(T(R(x))))$ (C) $R(T(U(S(x))))$
(D) $S(T(R(U(x))))$ (E) none of these

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49. Let $P(x)$ be a polynomial. If the remainder when $P(x)$ is divided by $x - 13$ is 37 and the remainder when $P(x)$ is divided by $x - 37$ is 13, then the remainder when $P(x)$ is divided by $(x - 13)(x - 37)$ is:

(A) $-x + 50$ (B) 50 (C) $x - 24$ (D) $x + 24$ (E) $2x + 50$

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50. Consider the graph of $y = x^2$. The number of points where this parabola intersects the circle of radius $2\sqrt{5}$ centered at $(-3, 3)$ is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Answer Key

1. B	18. C	35. C
2. B	19. A	36. E
3. A	20. E	37. A
4. D	21. C	38. D
5. A	22. E	39. B
6. A	23. A	40. D
7. D	24. A	41. D
8. E	25. B	42. B
9. C	26. A	43. B
10. C	27. C	44. D
11. A	28. C	45. D
12. C	29. B	46. D
13. E	30. E	47. C
14. B	31. C	48. E
15. B	32. E	49. A
16. B	33. E	50. D
17. D	34. E	