Mathematics Competition

Indiana University of Pennsylvania

2005

Directions:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. Define the cardinality of a finite set A, denoted by $n(A)$, to be the number of elements in the set A. If A and B are finite sets, with $n(A) = 7$, $n(A \cap B) = 5$, and $n(A \cup B) = 10$, then $n(B)$ is equal to:

(A) 10 (B) 8 (C) 1 (D) 2 (E) 14

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2. The effective resistance R of two resistors R_1 and R_2 that are connected in parallel is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

If R_1 is 5 ohms and R is 3 ohms, then the value of R_2 in ohms is:

(A) 2 (B) 7.5 (C) 8 (D) 10.5 (E) 15

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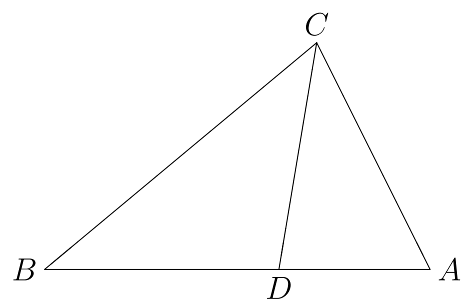
3. In the figure below, A, B, and D are collinear. Suppose $BD \cong CD$ and $\mathrm{m}\angle CBA = 40°$. The measure of $\angle CDA$ is:

(A) 40°
(B) 60°
(C) 80°
(D) 100°
(E) 120°



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4. A person drives from home to a destination at 60 miles per hour. The return trip is driven at 30 miles per hour. The average speed during the total trip is:

(A) 40 mph (B) 42 mph (C) 45 mph (D) 48 mph (E) 50 mph

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5. If $\tan\theta = 3/2$ and $\cos\theta < 0$, then $\sec\theta$ is equal to:

(A) $\frac{2}{3}$ (B) $\frac{\sqrt{13}}{2}$ (C) $\frac{-\sqrt{13}}{2}$ (D) $\frac{2\sqrt{13}}{3}$ (E) $\frac{-2\sqrt{13}}{3}$

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6. For any real number x, define $f(x) = x^2 - 4x$. Then $f(t-3)$ is equal to:

(A) $t^2 - 4t - 12$ (B) $t^2 - 4t - 3$ (C) $t^2 - 6t - 9$ (D) $t^2 - 10t - 9$
(E) $t^2 - 10t + 21$

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7. The side of one square is 5 units longer than that of another square. Their areas differ by 105 square units. The lengths of their sides are:

(A) 5 and 10 units (B) 7.5 and 12.5 units (C) 8 and 13 units
(D) 15 and 20 units (E) none of these

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8. A term that appears in the simplification of $(4\sqrt{3}-\sqrt{6})(5\sqrt{3}+2\sqrt{6})$ is:

(A) $9\sqrt{2}$ (B) $-2\sqrt{6}$ (C) $16\sqrt{9}$ (D) $-5\sqrt{18}$ (E) -12

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9. A certain football team wins 80% of its games when the temperature is below freezing. It wins 60% of its games if the temperature is not below freezing. If the probability is 2/3 that the temperature will be below freezing for its next game, the probability that the team will win that game is:

(A) $\frac{11}{15}$ (B) $\frac{1}{2}$ (C) $\frac{3}{5}$ (D) $\frac{7}{12}$ (E) $\frac{13}{15}$

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10. If $A = P(1+r)^t(1+s)^u$ where all of the variables represent positive quantities, then

$$\frac{\ln(A/P)}{\ln(1+r)\ln(1+s)}$$

is equal to:

(A) $(1+r)t+(1+s)u$ (B) $(1+s)t+(1+r)u$ (C) $st+ru$
(D) $\dfrac{t}{\ln(1+r)}+\dfrac{u}{\ln(1+s)}$ (E) $\dfrac{t}{\ln(1+s)}+\dfrac{u}{\ln(1+r)}$

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11. An expression that is **not** equivalent to $\left|\dfrac{|x-5|}{2}\right|$ is:

(A) $\left|\dfrac{|x+5|}{-2}\right|$ (B) $\left|\dfrac{|x-5|}{-2}\right|$ (C) $\left|\dfrac{|-x+5|}{2}\right|$ (D) $\left|\dfrac{|5-x|}{2}\right|$
(E) none of these

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12. John is driving home from the store. He drives four miles east on Elm St. then turns left on Park Ave. and drives three miles. Next, John turns right on Spruce Lane and drives two miles. Finally, John turns left on Main St. and drives five miles. If Superman flies the direct route from the store to John's house, the number of miles he flies is:

(A) 20 (B) 10 (C) 14 (D) 7 (E) $5+\sqrt{29}$

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13. The coefficient of x^5y^2 in the expansion of $(x + 2y)^7$ is:

(A) 7 (B) 84 (C) 48 (D) 21 (E) 42

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14. If $x = b(\sin\theta - \cos\theta)$, then $\dfrac{2bx}{x^2 - b^2}$ is equivalent to:

(A) $\sin\theta$ (B) $\cos\theta$ (C) $\csc\theta + \sec\theta$ (D) $\csc\theta - \sec\theta$
(E) none of these

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15. Sharon runs a single lap on a track in 100 seconds. Jessica runs the same lap in 90 seconds. If both start at the same time from the same point on the track and each maintains a constant rate, the number of minutes before Jessica laps Sharon is:

(A) 9 (B) 11 (C) 13.5 (D) 15 (E) 18.5

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16. The first five rows of a Pascal-like array of positive integers are shown. The sum of the numbers in the tenth row would be equal to:

(A) 127
(B) 255
(C) 511
(D) 1023
(E) 2047

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 2 & & & \\
 & & 1 & & 3 & & 3 & & \\
 & 1 & & 4 & & 6 & & 4 & \\
1 & & 5 & & 10 & & 10 & & 5 \\
 & & & & \vdots & & & &
\end{array}$$

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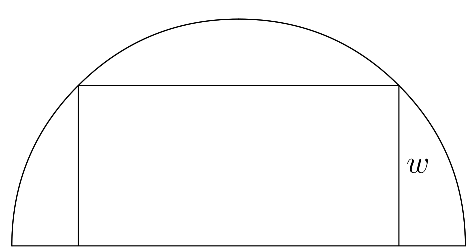
17. In the figure, a rectangle of height w is inscribed in a semicircle of diameter 8. The area of the rectangle is:

(A) $2w\sqrt{2}$
(B) w^2
(C) $2w^2$
(D) $2w\sqrt{16 - w^2}$
(E) $2w\sqrt{32 - 2w^2}$



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18. The number of distinct real solutions to the equation

$$|x^2 + 6x + 7| = 2$$

is:

(A) 1 (B) 2 (C) 3 (D) 4 (E) none of these

---

19. If $f(x) = 2x + 3$ and $g(x) = x^2 + 4x + 1$, then $(f \circ g - g \circ f)(x)$ is equal to:

(A) 0 (B) 1 (C) $2x^2 + 8x + 5$ (D) $-4x^2 + 12x + 9$
(E) $-2x^2 - 12x - 17$

---

20. A number equal to

$$\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$$

is:

(A) 2 (B) 3 (C) $2\sqrt{3}$ (D) $3\sqrt{2}$ (E) $3\sqrt{2} - 2\sqrt{3}$

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21. If

$$5 = \frac{\ln(1 + y)}{\ln(1 + x)}$$

where $x > -1$ and $y > -1$, then y can be written as a polynomial in x. The coefficient of x^4 in this polynomial is:

(A) 1 (B) 5 (C) 10 (D) 15 (E) none of these

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22. Point P has coordinates $(5, -4)$ and point Q has coordinates $(-3, -2)$. The equation for the perpendicular bisector of $\overline{PQ}$ is:

(A) $y = -4x + 1$ (B) $y = 4x - 7$ (C) $y = -4x + 16$ (D) $y = \frac{-1}{4}x - \frac{11}{4}$
(E) $y = \frac{1}{4}x - \frac{5}{4}$

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23. A teacher is buying supplies for the school. Using the tax-exempt status, she buys highlighters for 50¢ each, three-ring binders for \$3 each, and hole punchers for \$10 each. She buys a total of 100 items and spent exactly \$100. If she bought at least one of each item, the number of hole punchers she bought was:

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

---

24. The number of sets S satisfying

$$\{a, b\} \subset S \subseteq \{a, b, c, d, e\}$$

is:

(A) 3 (B) 6 (C) 7 (D) 8 (E) none of these

---

25. Suppose a person has two favorite numbers a and b and her favorite polynomial is $2x^3 - ax + b$. Also, two factors of her favorite polynomial are $x + 2$ and $x - 1$. Then the value of $a + b$ is:

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

---

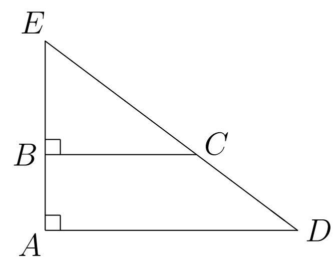
26. If $\pi/2 < x < 3\pi/2$, then $\sin(2x)$ equals:

(A) $\sin(x - \pi/2)$ (B) $\sin(\pi/2 - x)$ (C) $\cos(2x)$ (D) $\cos(3x)$
(E) $\sin(\pi - 2x)$

---

27. In the figure below, $AB = 2$, $BE = 3$, and $BC = 4$. The length of $\overline{CD}$ is:

(A) 2
(B) 3
(C) $3\frac{1}{3}$
(D) 4
(E) 5



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28. We start with an acid solution of unknown strength. When one ounce of water is added to the mixture, the new mixture is 20% acid. When one ounce of acid is added to the new mixture, the result is $33\frac{1}{3}$% acid. The percentage of acid in the original mixture was:

(A) 22% (B) 24% (C) 25% (D) 30% (E) $33\frac{1}{3}$%

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29. The domain of the function defined by

$$f(x) = \left(1 - \frac{3}{x+2}\right)^{1/4}$$

expressed using interval notation is:

(A) $(-\infty, -2) \cup [1, \infty)$ (B) $(-\infty, -2) \cup (1, \infty)$ (C) $(-\infty, -2) \cup (2, \infty)$
(D) $(-2, 1]$ (E) none of these

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30. If $x + 2y + 3z = 6$, $2x - 3y + 2z = 14$, and $3x + y - z = -2$, then $x + y + z$ is equal to:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

---

31. The sum of the solutions to the equation

$$3^{2x} - 8 \cdot 3^{x} + 12 = 0$$

is:

(A) $\log_2 3$ (B) $\log_3 2$ (C) $\log_3 6$ (D) $\log_2 8$ (E) $\log_3 12$

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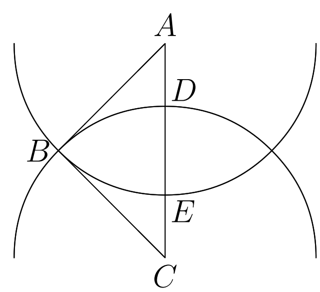
32. In the figure, $\triangle ABC$ is an isosceles right triangle with $AB = BC = 2$. A circular arc of radius 2 with center C meets the hypotenuse at D, and a circular arc of radius 2 with center A meets the hypotenuse at E. To the nearest tenth of a square unit, the combined area of the two shaded regions is:



(A) 0.9
(B) 1.0
(C) 1.1
(D) 1.2
(E) 1.3

---

33. A couple has nine daughters. The sum of the daughters' ages is 207. The age difference between consecutive daughters is always three years. The age of the middle daughter is:

(A) 21 (B) 23 (C) 25 (D) 27 (E) 29

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34. Five cards, each having a letter on one side and a number on the other side, are laid out side by side as illustrated below. A sign reads, "If there is a D on one side of the card, then there is a 5 on the other side of the card." The smallest number of cards that must be turned over to determine whether the sign is correct is:

$\boxed{2}$ $\boxed{\text{D}}$ $\boxed{\text{X}}$ $\boxed{5}$ $\boxed{3}$

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

---

35. The positive value of x that is equal to the infinite repeated fraction

$$x = \cfrac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}$$

is:

(A) $\dfrac{-1+\sqrt{3}}{2}$ (B) $\dfrac{1}{2}$ (C) $\dfrac{1+\sqrt{5}}{2}$ (D) $\dfrac{-1+\sqrt{5}}{2}$ (E) $\dfrac{1+\sqrt{3}}{2}$

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36. If $i^2 = -1$, then the expression

$$\cos 45° + i \cos 135° + \cdots + i^n \cos(45° + 90°n) + \cdots + i^{40} \cos 3645°$$

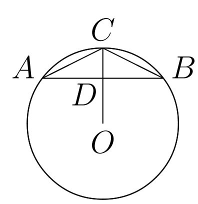
is equal to:

(A) $\dfrac{\sqrt{2}}{2}$ (B) $-10i\sqrt{2}$ (C) $\dfrac{21\sqrt{2}}{2}$ (D) $\dfrac{\sqrt{2}}{2}(21-20i)$

(E) none of these

---

37. In the figure, the radius of the circle is 1 cm. The chord $\overline{AB}$ and the radius $\overline{OC}$ are perpendicular, meeting at the point D. If the length of $\overline{OD}$ is x cm, then the area of $\triangle ABC$ in cm^2 is:



(A) $(1-x)\sqrt{1-x^2}$
(B) $x\sqrt{1-x^2}$
(C) $\sqrt{1-x^2}$
(D) $x(1-x)$
(E) 1

---

38. Sean can mow a lawn in 28 minutes. Bill can get the same job done in 24 minutes. They start working together, but after a while Sean quits. Bill finishes the job by working as much additional time as he had just worked with Sean helping. The number of minutes that Bill worked was:

(A) $11\frac{3}{7}$ (B) $13\frac{1}{3}$ (C) 15 (D) $16\frac{4}{5}$ (E) $17\frac{14}{15}$

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39. Consider the finite universal set $U = \{1, 2, 3, 4\}$. For a subset A of U, define the function

$$f(A) = \begin{cases} A & \text{if } n(A) \geq \frac{1}{2}n(U) \\ A^C & \text{if } n(A) < \frac{1}{2}n(U) \end{cases},$$

where A^C denotes the complement of A in U and $n(A)$ denotes the number of elements of A. The number of subsets A of U that satisfy $f(A) = f(A^C)$ is:

(A) 1 (B) 4 (C) 6 (D) 8 (E) none of these

---

40. The solution set to the inequality $1 < x + 2 < 2x + 5$ is:

(A) $(-1, \infty)$ (B) $(-3, \infty)$ (C) $(-3, -1)$ (D) $(-\infty, -3) \cup (-1, \infty)$
(E) none of these

---

41. The solution set to the equation

$$\log_2(x^{5+\log_2 x}) = 6$$

is:

(A) $\{0\}$ (B) $\{2\}$ (C) $\{2, 3\}$ (D) $\{1, -6\}$ (E) none of these

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42. If the line $y = mx + 2$ intesects the circle $x^2 + y^2 = 1$ exactly once, then the value(s) of m is(are):

(A) 2 (B) ± 2 (C) $\pm 3/2$ (D) $\pm\sqrt{2}$ (E) $\pm\sqrt{3}$

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43. The difference between the largest and smallest roots of

$$30x^3 - 31x^2 - 66x + 72 = 0$$

is:

(A) $\frac{-3}{2}$ (B) $\frac{6}{5}$ (C) $\frac{4}{3}$ (D) $\frac{17}{6}$ (E) $\frac{27}{10}$

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44. If $21 \sec x = 20 \csc x$, the value of $|\cos x| - |\sin x|$ is equal to:

(A) $\frac{20}{21}$ (B) $\frac{21}{20}$ (C) $\frac{41}{29}$ (D) $\frac{1}{29}$ (E) none of these

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45. Every person in a certain town either tells the truth all of the time or lies all of the time. You encounter three of these townsmen. The first says, "All three of us are liars." The second says, "Exactly two of us are liars." The third says, "The other two are liars." The townsmen you can trust are:

(A) only the first (B) only the second (C) only the third
(D) the second and third (E) none of these

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46. If (x, y) is the ordered pair of real numbers that satisfies the system

$$\begin{aligned} \sqrt{x+y} &= x+3y-7 \\ \sqrt{x+3y} &= x+y-1, \end{aligned}$$

then y is equal to:

(A) $1/2$ (B) $3/2$ (C) $5/2$ (D) 3 (E) none of these

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47. For $x > 0$ and $c > 0$, if $y = 5\ln(x^2 + cx)$, then $2x + c$ equals:

(A) $\sqrt{c^2 - 4e^{y/5}}$ (B) $\sqrt{c^2 + 4e^{y/5}}$ (C) $\sqrt{4e^{y/5} - c^2}$ (D) $\sqrt{4e^{-y/5} - c^2}$
(E) none of these

---

48. The number of positive integers n for which the inequality

$$2n < n^2 < 2^n < n! < n^n$$

is false is:

(A) 4 (B) 3 (C) 2 (D) 1 (E) none of these

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49. The solution set to the equation

$$\sqrt{x - 2\sqrt{x-1}} + \sqrt{x + 24 - 10\sqrt{x-1}} = 4$$

is:

(A) $\{2, 5\}$ (B) $(-\infty, \infty)$ (C) $[1, \infty)$ (D) $[2, \infty)$ (E) none of these

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50. Consider a quadrilateral $ABCD$ with $\angle ABC = 45°$ and $\angle BCD = 90°$. If $|AB| = 4$, $|BC| = 3\sqrt{2}$, and $|CD| = \sqrt{6}$, then $|AD|$ is equal to:

(A) $2\sqrt{2}$ (B) $2\sqrt{3} - 2$ (C) $6 - 2\sqrt{3}$ (D) $3\sqrt{2} - \sqrt{6}$
(E) none of these

Answer Key

1. B
2. B
3. C
4. A
5. C
6. E
7. C
8. A
9. A
10. E
11. A
12. B
13. B
14. D
15. D
16. D
17. D
18. C
19. E
20. A
21. B
22. B
23. E
24. C
25. D
26. E
27. C
28. C
29. A
30. C
31. E
32. A
33. B
34. C
35. D
36. D
37. A
38. D
39. E
40. A
41. E
42. E
43. D
44. D
45. B
46. C
47. B
48. A
49. E
50. B