Mathematics Competition
Indiana University of Pennsylvania
2006

Directions:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. The set of all possible quadrants in which (x, y) can be located if $xy > 0$ is:

(A) $\{I, II\}$ (B) $\{I\}$ (C) $\{II, III\}$ (D) $\{II, IV\}$ (E) $\{I, III\}$

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2. Ara and Shea were once the same height. Since then Shea has grown 20%, while Ara has grown half as many inches as Shea. Shea is now 60 inches tall. Ara's height, in inches, is:

(A) 48 (B) 51 (C) 52 (D) 54 (E) 55

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3. Because of the relative strengths of baseball teams A and B, it is estimated that the probability of A winning an individual game with B is $5/9$, while the probability of B winning an individual game with A is $4/9$. If A and B are playing the World Series (best of seven) which currently stands in favor of A three games to one, then the probability that A will win the series without a game seven being necessary is:

(A) $20/81$ (B) $65/81$ (C) $7/9$ (D) $2/3$ (E) $5/9$

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4. Let $f(x) = 2x^2 + 1$, and assume $h \neq 0$. Then $\dfrac{f(x+h) - f(x)}{h}$ is equal to:

(A) $2x + h$ (B) $\dfrac{4xh + 2h^2 + 2}{h}$ (C) $4x + 2h$ (D) $\dfrac{x^2 + 2xh + h^2}{h}$
(E) $2h$

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5. If $\log_b 2 = a$, then $\log_b(64b)$ is equal to:

(A) a (B) $6a$ (C) $6a - 1$ (D) $6a + 1$ (E) $a + 6$

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6. If $3^a = 2$ and $7^b = 5$, then $9^{4a} \times 49^{2b}$ is equal to:

(A) $2^4 \times 5^2$ (B) $2^6 \times 5^6$ (C) $2^8 \times 5^4$ (D) $2^4 \times 5^8$ (E) $2^6 \times 5^4$

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7. Of the following points, the one(s) that lie(s) on the line that contains the point $(2, -3)$ and has slope $-7/4$ is(are):

(A) $(4, -13/2)$ (B) $(-2, 4)$ (C) $(0, 1/2)$ (D) all of these
(E) none of these

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8. If $t > 0$, then the equation $(1 + x)^t = (1 + \frac{.04}{2})^{2t}$ has the solution:

(A) $x = .0404$ (B) $x = .0402$ (C) $x = .04$ (D) $x = .0204$
(E) $x = .02$

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9. Let $A = \{1, 2, 3, 4, 5, 7, 8, 10, 11, 14, 17, 18\}$. The number of subsets of A that have exactly three elements is:

(A) 220 (B) 440 (C) 816 (D) 1320 (E) 1728

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10. The expression $\csc^2\theta \tan\theta \sin\theta$ is equal to:

(A) $\tan\theta$ (B) $\csc\theta$ (C) $\sin\theta$ (D) $\sec\theta$ (E) $\cos\theta$

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11. A man and his son are 6 miles apart. They both are walking 3 miles per hour toward each other. Their dog runs at a speed of 20 miles per hour back and forth between the father and his son until they meet. The total distance the dog runs is:

(A) 10 miles (B) 20 miles (C) 40 miles (D) 60 miles
(E) an infinite distance

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12. The solution to the equation $3^{z+3} = 2^z$ is:

(A) $\dfrac{\ln 3}{\ln 3 - \ln 2}$ (B) $\dfrac{-2\ln 3}{\ln 3 - \ln 2}$ (C) $\dfrac{3\ln 3}{\ln 3 - \ln 2}$ (D) $\dfrac{-3\ln 3}{\ln 3 - \ln 2}$
(E) $\dfrac{-3\ln 2}{\ln 3 - \ln 2}$

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13. It took me awhile to learn how to flip flapjacks. In fact, if $a + \dfrac{1}{a} = 6$, then the number of attempts it took me was $a^2 + \dfrac{1}{a^2}$, which is:

(A) 34 (B) 35 (C) 36 (D) 38 (E) 42

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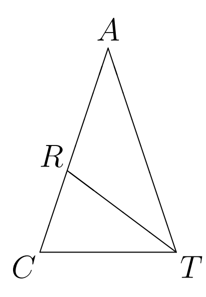
14. In $\triangle CAT$, we have $m\angle ACT = m\angle ATC$ and $m\angle CAT = 36°$. If $\overline{TR}$ bisects $\angle ATC$, then $m\angle CRT$ is equal to:



(A) 36°
(B) 54°
(C) 72°
(D) 90°
(E) 108°

---

15. The expression $\sqrt{18x} - \sqrt{2x}$ is equal to:

(A) 3 (B) $4x$ (C) $4\sqrt{x}$ (D) $2\sqrt{2x}$ (E) none of these

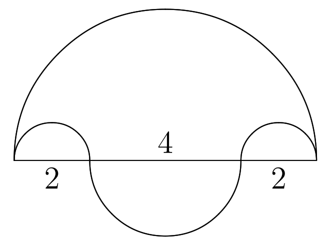
16. Bob can run one lap around a track in 60 seconds. Joe can run one lap in 50 seconds. Mike can run one lap in 40 seconds. If they all begin running from the starting line, the time it will take for all three of them to be together again is:

(A) 1500 sec (B) 1200 sec (C) 600 sec (D) 240 sec (E) never

---

17. If all of the arcs in the figure below are semicircles and the lengths of the line segments are as indicated, then the area of the shaded region is:

(A) 9π
(B) 16π
(C) 17π
(D) 64π
(E) 65π



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18. The sum of the roots of the equation

$$x^3 - 5x^2 + 4 = 0$$

is:

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

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19. The sum $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{8\cdot 9} + \dfrac{1}{9\cdot 10}$ is equal to:

(A) $\dfrac{6}{7}$ (B) $\dfrac{7}{8}$ (C) $\dfrac{8}{9}$ (D) $\dfrac{9}{10}$ (E) 1

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20. The solution set to the inequality

$$x^3 + 8x^2 + 16x < 0$$

is:

(A) $\emptyset$ (B) $(-\infty, -4)$ (C) $(-\infty, 0)$ (D) $(-4, 0)$ (E) none of these

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21. If

$$\log_2(\log_5 x) = \log_3(\log_2 y) = 1,$$

then $x + y$ is equal to:

(A) 41 (B) 33 (C) 7 (D) 5 (E) none of these

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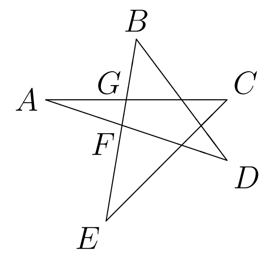
22. If in the figure, $m\angle A = 20°$ and $m\angle AFG = m\angle AGF$, then $m\angle B + m\angle D$ is equal to:



(A) 48°
(B) 60°
(C) 72°
(D) 80°
(E) 40°

---

23. The number of real solutions to the equation

$$|x^2 + 6x + 1| = 8$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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24. If the function f is defined by

$$f(x) = 8 - x^3,$$

then $f^{-1}(x)$ is given by:

(A) $x^3 - 8$ (B) $\dfrac{1}{8 - x^3}$ (C) $2 - \sqrt[3]{x}$ (D) $\sqrt[3]{x + 8}$ (E) none of these

---

25. If $x + y$ is to $4x - 3y$ as 2 is to 3 and $y \neq 0$, then x/y is equal to:

(A) 3/2 (B) 5/9 (C) 9/5 (D) 3/5 (E) 5/2

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26. Of the following pairs of equations, the pair with the most points of intersection of their graphs is:

(A) $y = x^2$ and $y = |x|$
(B) $y = \pm x$ and $y = 4 - x^2$
(C) $y = \sqrt[3]{x}$ and $y = x^3$
(D) $y = e^x$ and $y = x^2 + x + 1$
(E) $y = \sin x$ and $y = \dfrac{2x}{5\pi}$

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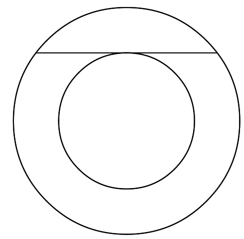
27. The area of the shaded ring given that the chord tangent to the inner circle has a length of 10 is:



(A) 100π
(B) 50π
(C) 25π
(D) 10π
(E) impossible to determine

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28. The solution to the inequality

$$\frac{(x-4)^3}{x\sqrt{x-2}} \leq 0$$

is:

(A) $x = 4$ (B) $2 < x \leq 4$ (C) $x > 4$ (D) $x \leq 4$ (E) $0 \leq x \leq 2$

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29. If three six-sided dice that are numbered in the usual way are rolled, then the probability of obtaining a sum of 16, 17, or 18 is:

(A) $\frac{5}{108}$ (B) $\frac{1}{36}$ (C) $\frac{3}{17}$ (D) $\frac{1}{54}$ (E) $\frac{1}{12}$

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30. Two people working together can complete a task in 2 hours. Working individually it takes one of them 3 hours longer than it does for the other to complete the task. The time it takes for the slower person to do the job alone is:

(A) 6 hrs (B) 5.5 hrs (C) 4.5 hrs (D) 3 hrs (E) none of these

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31. The expression

$$\left(\log_{1/2}\tfrac{1}{3}\right)\left(\log_{1/3}\tfrac{1}{4}\right)\left(\log_{1/4}\tfrac{1}{5}\right)\left(\log_{1/5}\tfrac{1}{6}\right)\left(\log_{1/6}\tfrac{1}{7}\right)$$

is equal to:

(A) $\log_{1/2} 7$ (B) $\log_{1/2} 3$ (C) $\log_2 4$ (D) $\log_3 5$ (E) $\frac{\ln 7}{\ln 2}$

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32. If $f(x) = \dfrac{x}{x+1}$, then $(f \circ f)^{-1}(x)$ is equal to:

(A) $\frac{x}{2x+1}$ (B) $\frac{-x}{x-1}$ (C) $\frac{-x}{2x-1}$ (D) $\frac{x}{2x-1}$ (E) $\frac{x}{x-1}$

---

33. If $(x + x^{-1})^2 = 4$, then $x^3 + x^{-3}$ could be:

(A) 2 (B) $9/2$ (C) $2\sqrt{2}$ (D) $5\sqrt{2}$ (E) none of these

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34. Let l be a line through the origin in the xy-plane that makes an angle of $30°$ with the positive x-axis. The x-coordinate of the point of intersection of the line l with the curve $x^3 + y^3 = 1$ is:

(A) $\sqrt[3]{\dfrac{27+3\sqrt{3}}{26}}$ (B) $\sqrt[3]{\dfrac{27-3\sqrt{3}}{26}}$ (C) $\dfrac{\sqrt[3]{4}}{2}$ (D) $\sqrt[3]{\dfrac{-1+3\sqrt{3}}{26}}$

(E) $\sqrt[3]{\dfrac{-1-3\sqrt{3}}{26}}$

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35. The solution to the inequality

$$|x-2| < 3|x+6|$$

is:

(A) $x < -10$ (B) $-10 < x < -4$ (C) $x > -4$ or $-10 < x < -4$
(D) $x < -4$ (E) $x < -10$ or $x > -4$

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36. The value of $\cos(\pi/64)$ is:

(A) $\dfrac{\sqrt{2}}{2}$ (B) $\dfrac{\sqrt{2+\sqrt{2}}}{2}$ (C) $\dfrac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$

(D) $\dfrac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}$ (E) $\dfrac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}{2}$

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37. The host of a game show lets you choose from among three doors behind exactly one of which is a prize. Once you have made your first choice, the host opens one of the other two doors which does not have the prize behind it. You then choose again from among the unopen doors to try to win the prize. The probability that you win the prize if you switch from the door that you originally picked is:

(A) $3/4$ (B) $2/3$ (C) $1/2$ (D) $1/3$ (E) none of these

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38. The solution set to the equation

$$\sqrt{5x-1}-\sqrt{x+2}=1$$

is contained in the interval:

(A) $(-1,1)$ (B) $(1,2)$ (C) $[2,3]$ (D) $(3,5]$ (E) none of these

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39. Let $\overline{AB}$ be a chord of a circle and let P be the point where the tangents to the circle at A and B meet. If $AB = 10$ and $AP = BP = 13$, then the radius of the circle is:

(A) 13 (B) 12 (C) $\dfrac{65}{12}$ (D) $\dfrac{60}{13}$ (E) none of these

---

40. The sum of the real roots of the equation

$$x^4+5x^3+8x^2-2x-12$$

lies in the interval:

(A) $(-\infty,-10)$ (B) $(-10,-4)$ (C) $(-4,0)$ (D) $(0,10)$ (E) $(10,\infty)$

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41. If $(a^b)^c = (d^e)^f = (g^h)^i$ with all of the lettered quantities greater than one, then the value of $\log_a g$ is:

(A) $\dfrac{bc}{ef}$ (B) $\dfrac{bc}{hi}$ (C) $\dfrac{ef}{hi}$ (D) $\dfrac{ef}{bc}$ (E) $\dfrac{hi}{ef}$

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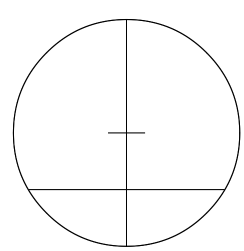
42. A cylindrical water tank lying on its side is filled to 1/4 its height as shown in the cross-section below. If the tank is 10 feet long with a diameter of 6 feet, the volume in cubic feet of water in the tank is:



(A) $120\pi - 90\sqrt{3}$
(B) $30\pi - 45\sqrt{3}/2$
(C) $45\pi/2$
(D) $45\sqrt{3}$
(E) none of these

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43. The number of positive real solutions to the equation

$$\sqrt[3]{x+4}+\sqrt{x}=4$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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44. The natural logarithm of the product of the solutions to the equation

$$6(\ln x)^3 - 11(\ln x)^2 + 6(\ln x) - 1 = 0$$

is:

(A) 11/6 (B) 3/2 (C) 1/6 (D) 6 (E) 2/3

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45. There are 100 prisoners in a circular formation. They are to be released one at a time beginning with #1 and releasing every other prisoner until all prisoners are released (i.e., #1, #3, #5, etc.). The last prisoner to be released is:

(A) #100 (B) #64 (C) #8 (D) #72 (E) none of these

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46. The sum of the solutions for x, y, and z to the system

$$\begin{aligned} \frac{1}{x}+\frac{1}{y}+\frac{1}{z} &= 12 \\ \frac{1}{x}+\frac{2}{y}+\frac{4}{z} &= 27 \\ \frac{2}{x}+\frac{3}{y}+\frac{4}{z} &= 35 \end{aligned}$$

is:

(A) $\frac{47}{60}$ (B) $\frac{17}{20}$ (C) $\frac{13}{15}$ (D) $\frac{1}{12}$ (E) 12

---

47. The number of solutions to the equation

$$4\cos^2\theta\sin\theta + 2\cos^2\theta + 2\cos\theta\sin\theta + \cos\theta - 2\sin\theta - 1 = 0$$

that lie in the interval $[0, 2\pi)$ is:

(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

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48. Each face of a regular tetrahedron is painted either red, white, or blue. Two such paintings create the same *pattern* if some rotation of the tetrahedron makes them look identical. The number of distinct patterns that are possible is:

(A) 12 (B) 24 (C) 64 (D) 81 (E) none of these

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49. If x, y, u, and v form a solution to the system

$$\begin{aligned} xy + x + y &= 9 \\ \sqrt{u} - 2\sqrt{v} &= 1 \\ x^2 + y^2 + 2(x + y) &= 27 \\ u - 4v &= 5, \end{aligned}$$

then the smallest that $x + y + u + v$ can be is:

(A) -12 (B) $-2\sqrt{3} + 2$ (C) -5 (D) 1 (E) $\sqrt{2} - 3$

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50. The following picture depicts a quadrilateral inscribed in a circle. The sides have lengths a, b, c, d and the diagonals have lengths x and y. In this situation, x is equal to:

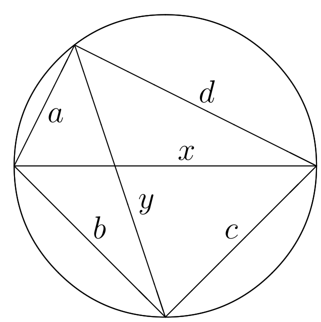
(A) $ac + bd$

(B) $\dfrac{ac + bd}{y}$

(C) $(ac)(bd)$

(D) $\dfrac{ab + cd}{y}$

(E) $ab + cd$



Answer Key

1. E
2. E
3. B
4. C
5. D
6. C
7. D
8. A
9. A
10. D
11. B
12. D
13. A
14. C
15. D
16. C
17. A
18. E
19. D
20. E
21. B
22. D
23. D
24. E
25. C
26. E
27. C
28. B
29. A
30. A
31. E
32. C
33. A
34. B
35. E
36. E
37. B
38. C
39. C
40. C
41. B
42. B
43. B
44. A
45. D
46. A
47. A
48. E
49. D
50. B