Mathematics Competition

Indiana University of Pennsylvania

2008

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. Two 6-sided dice are rolled. The probability that the sum of the two numbers on the dice is greater than 10 is:

(A) 1/18 (B) 1/12 (C) 5/18 (D) 3 (E) none of these

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2. A piece of string 15 inches long is cut into four pieces. If each successive piece is half as long as the preceding piece, then the length of the first piece, in inches, is:

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

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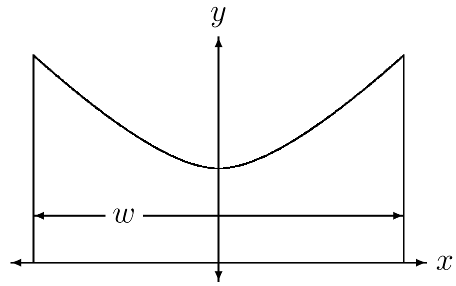
3. The roof of a building is in the shape of the hyperbola $y^2 - x^2 = 25$, where x and y are in meters. The walls are 11 meters high. The distance w, in meters, between the outside walls is:



(A) 96
(B) $8\sqrt{6}$
(C) $4\sqrt{6}$
(D) $\sqrt{146}$
(E) none of these

---

4. Of the following numbers, the only one that is a solution of the equation

$$x^2 + x^{-2} = 3$$

is:

(A) $\sqrt{\frac{1+\sqrt{5}}{2}}$ (B) $\sqrt{\frac{2+\sqrt{5}}{2}}$ (C) $\sqrt{\frac{3+\sqrt{5}}{2}}$ (D) $\sqrt{\frac{4+\sqrt{5}}{2}}$
(E) $\sqrt{\frac{5+\sqrt{5}}{2}}$

---

5. Expressed as a single logarithm,

$$\frac{3}{2}\ln(4x^8) - \frac{4}{5}\ln(2m^5)$$

is equivalent to:

(A) $\ln\left(\frac{2^{11/5}x^{12}}{m^4}\right)$ (B) $\ln\left(\frac{2^{5/11}x^{12}}{m^4}\right)$ (C) $\ln\left(\frac{2^{11/5}x^4}{m}\right)$
(D) $\frac{7}{10}\ln(8xm^{40})$ (E) $8\ln(xm^{40})$

---

6. The number of ordered pair solutions (x, y) with real coordinates of the system

$$\begin{aligned} 3x^2 - y^2 &= 3 \\ 5x^2 - 2y^2 &= 2 \end{aligned}$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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7. The area of a triangle, in cm^2, whose sides have length 6 cm, 7 cm, and 9 cm is:

(A) $2\sqrt{110}$ (B) $12\sqrt{3}$ (C) $9\sqrt{6}$ (D) 22 (E) none of these

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8. An interval that contains all real solutions to the equation

$$\frac{16}{x^2+4x+16} = \frac{x}{x-4} - \frac{64}{x^3-64}$$

is:

(A) $(-\infty, -2)$ (B) $(-10, 0)$ (C) $(-5, 5)$ (D) $(0, \infty)$
(E) none of these

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9. The sum

$$1000(1.05)^{10} + 1000(1.05)^9 + \cdots + 1000(1.05) + 1000$$

is equal to:

(A) $1000 \cdot \dfrac{1.05^7 - 1}{1.05 - 1}$ (B) $1000 \cdot \dfrac{1.05^8 - 1}{1.05 - 1}$ (C) $1000 \cdot \dfrac{1.05^9 - 1}{1.05 - 1}$

(D) $1000 \cdot \dfrac{1.05^{10} - 1}{1.05 - 1}$ (E) $1000 \cdot \dfrac{1.05^{11} - 1}{1.05 - 1}$

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10. The solution set for the equation

$$\cos 2\theta = 2 - 3\sin\theta$$

with $0 \le \theta \le 2\pi$ is:

(A) $\{\pi/6\}$ (B) $\{\pi/6, \pi/2\}$ (C) $\{\pi/6, 5\pi/6\}$ (D) $\{\pi/6, \pi/2, 5\pi/6\}$
(E) none of these

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11. If $x^2 = 3x + 4$, then x^5 is equal to:

(A) $3x + 4$ (B) $13x + 12$ (C) $51x + 52$ (D) $205x + 204$
(E) $819x + 820$

---

12. Suppose $A = 0, B = 1, \ldots, Z = 25$ are the "digits" in a base 26 left-to-right numeration system. Then $CBA - BNM$ is equal to:

(A) YES (B) NO (C) ANN (D) OK (E) none of these

---

13. The solution set to the equation

$$|3x + 1| = |x - 5|$$

is:

(A) $\{-3\}$ (B) $\{2\}$ (C) $\{-3, 2\}$ (D) $\{-3, 1\}$ (E) none of these

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14. A railroad tunnel is shaped like a semi-ellipse. The height of the tunnel at the center is 42 ft and the vertical clearance must be 28 ft at a point 25 ft from the center. An equation for the ellipse of which the tunnel is the top half is:

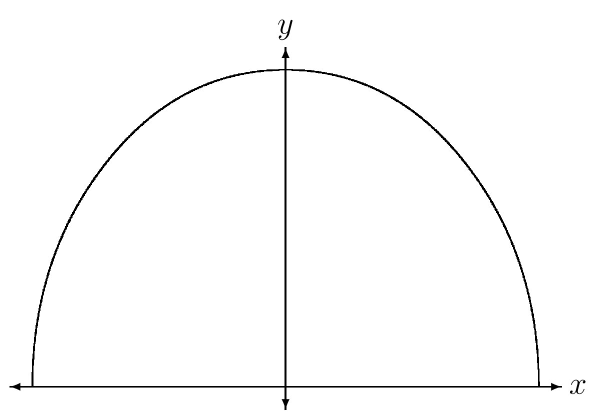
(A) $\dfrac{x^2}{1125} + \dfrac{y^2}{1764} = 1$

(B) $\dfrac{x^2}{625} + \dfrac{y^2}{1764} = 1$

(C) $\dfrac{x^2}{1764} + \dfrac{y^2}{1125} = 1$

(D) $\dfrac{x^2}{1125} + \dfrac{y^2}{784} = 1$

(E) none of these



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15. Suppose p is prime and $x^2 + ax + p$ has two integer roots. Then a must be:

(A) $\pm p$ (B) $p \pm 1$ (C) $1 \pm p$ (D) $\pm(p + 1)$ (E) none of these

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16. The solution set to the equation

$$2\log_{10} x - \log_{10} 3 = \log_{10}(x+6)$$

is:

(A) $\{18\}$ (B) $\{9\}$ (C) $\{6\}$ (D) $\{-3, 6\}$ (E) none of these

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17. A drawer contains 4 gray socks, 6 black socks and 10 brown socks. If two socks are randomly pulled out of the drawer in the dark, the probability that these socks are the same color is:

(A) 1/6 (B) 1/3 (C) 33/95 (D) 43/105 (E) none of these

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18. The solution set, in interval notation, of the inequality

$$\frac{-2(x^2+9)(x^3+27)}{x(5-x)^2} \geq 0$$

is:

(A) $[-3, 0)$ (B) $(-\infty, -3] \cup (0, 5)$ (C) $(-3, 0) \cup [5, \infty)$
(D) $(-\infty, -3) \cup [0, 5]$ (E) $(-\infty, \infty)$

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19. A water lily on a rigid stem growing vertically upward from the bottom of a pond extends 10 feet above the water's surface. At night it tilts until its top floats on the surface of the water 20 feet from the point where the vertical plant broke through the surface. The depth of water in the pond, in feet, is:

(A) 30 (B) 25 (C) 15 (D) 5 (E) none of these

---

20. In the binomial expansion of $(5x+2y)^4$, the coefficient of xy^3 is:

(A) 1000 (B) 16 (C) 160 (D) 600 (E) 625

---

21. The value of

$$(\log_2 4)(\log_8 16)(\log_{32} 64)(\log_{128} 256)(\log_{512} 1024)(\log_{2048} 4096)$$

is:

(A) $\frac{8}{3}$ (B) $\frac{16}{15}$ (C) $\frac{128}{35}$ (D) $\frac{256}{63}$ (E) $\frac{1024}{231}$

---

22. A circle is inscribed in a triangle with sides of lengths 3 cm, 4 cm, and 5 cm. The radius of this circle, in cm, is:

(A) $5/7$ (B) 1 (C) $7/5$ (D) 2 (E) none of these

---

23. The solution set for the equation

$$\sqrt{x+6}+\sqrt{2-x}=4$$

is:

(A) $\{-2\}$ (B) $\{-2,2\}$ (C) $\{0,2\}$ (D) $\{-2,1\}$ (E) none of these

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24. If $f(x)=1+\dfrac{1}{x}$, then the only false statement among the following is:

(A) $f(f(x))=\dfrac{2x+1}{x+1}$ (B) $f(f(f(x)))=\dfrac{3x+2}{2x+1}$

(C) $f(f(f(f(x))))=\dfrac{5x+3}{3x+2}$ (D) $f(f(f(f(f(x)))))=\dfrac{8x+5}{5x+3}$

(E) $f(f(f(f(f(f(x))))))=\dfrac{13x+7}{8x+5}$

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25. From a 10 liter container of 5% acid solution, a chemist drains x liters of the solution and replaces those x liters with x liters of a 2% acid solution to obtain an overall solution that is 4% acid. Then x is equal to:

(A) $5/2$ (B) $10/3$ (C) 4 (D) $4/5$ (E) 5

---

26. The solution in $(0,\pi)$ to the system

$$\begin{aligned} 4\sin^2 x &= 2-\sqrt{3} \\ 4\cos^2 x &= 2+\sqrt{3} \end{aligned}$$

is:

(A) $\pi/3$ (B) $\pi/4$ (C) $\pi/6$ (D) $\pi/12$ (E) $2\pi/5$

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27. Three players A, B, and C are seated around a table taking turns rolling a die. Player A rolls first, followed by B, and then C. Once a player has rolled a 6, the game is stopped and that player is declared the winner. If no 6 has been obtained after each of A, B, and C have rolled the die once, player A gets to roll again, followed by B, etc. The probability that player A wins this game is:

(A) $\frac{1}{6}$ (B) $\frac{1}{216}$ (C) $\frac{36}{91}$ (D) $\frac{25}{216}$ (E) none of these

---

28. The solution set for the inequality

$$\frac{2x}{x^2-25}+\frac{x}{x^2+x-30}\geq\frac{3x}{x^2+11x+30}$$

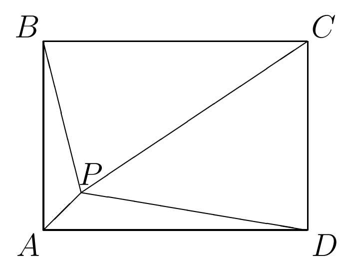
is:

(A) $(-\infty,-6)\cup(-5,0]\cup(5,\infty)$
(B) $(-6,-5)\cup[0,5)$
(C) $(-\infty,-6)\cup(-5,1]\cup(5,\infty)$
(D) $(-\infty,-6)\cup(-5,5)\cup[6,\infty)$
(E) none of these

---

29. In rectangle $ABCD$ below, $PA = 1$, $PB = 4$, and $PC = 7$. Then PD is equal to:

(A) $\sqrt{8}$
(B) $\sqrt{14}$
(C) $\sqrt{26}$
(D) $\sqrt{34}$
(E) $\sqrt{44}$



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30. The number of distinct real solutions of the equation

$$x^2-5x+(5-x)\sqrt{x}=0$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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31. The solution set to the equation

$$\ln\sqrt[4]{x}=\sqrt[3]{\ln x}$$

is:

(A) $\{1\}$ (B) $\{e^4\}$ (C) $\{e^8\}$ (D) $\{1,e^8\}$ (E) none of these

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32. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and let $\emptyset$ be the empty set. The power set of a set X is the set of all subsets of X and is denoted by $\mathcal{P}(X)$. An example of a set A such that $A \cap \mathcal{P}(A) \neq \emptyset$ is:

(A) $\mathbb{N}$ (B) $\mathbb{N} \cap \mathcal{P}(\mathbb{N})$ (C) $\mathbb{N} \cup \mathcal{P}(\mathbb{N})$ (D) $\emptyset$ (E) none of these

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33. The sum of the solutions of the equation

$$x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$$

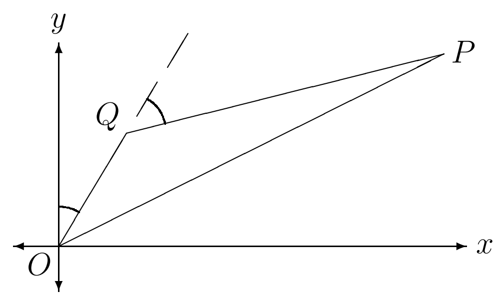
is:

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

---

34. In the figure below, $\overline{OQ}$ makes a $30°$ angle with the y-axis and $\overline{QP}$ makes a $45°$ angle with the extension of $\overline{OQ}$. If $OQ = 4$ and $QP = 8$, then the y-coordinate of P is:

(A) $2 + 2\sqrt{6} + 2\sqrt{2}$
(B) $2 + 2\sqrt{6} - 2\sqrt{2}$
(C) $2\sqrt{3} + 2\sqrt{6} + 2\sqrt{2}$
(D) $2\sqrt{3} - 2\sqrt{6} + 2\sqrt{2}$
(E) $2\sqrt{3} + 2\sqrt{6} - 2\sqrt{2}$



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35. The real solution of the equation

$$\sqrt[3]{x} + \sqrt{x} = 12$$

lies in the interval:

(A) $[0, 30]$ (B) $[30, 60]$ (C) $[60, 90]$ (D) $[90, 120]$ (E) $[120, 150]$

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36. A searchlight is shaped like a paraboloid. If the light source is located 3 feet from the base along the axis of symmetry and the opening is 8 feet across, then the depth of the searchlight, in feet, is:

(A) $4\sqrt{3}$ (B) $4/3$ (C) $16/3$ (D) 4 (E) none of these

---

37. The solution set to the equation

$$x^{\log_2 x} = 8x^2$$

is:

(A) $\{0, 8\}$ (B) $\{1, 3\}$ (C) $\{3\}$ (D) $\{8\}$ (E) none of these

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38. The value of $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$ is:

(A) $1+\sqrt{2}$ (B) $\sqrt{8}$ (C) $2-\sqrt{2}$ (D) 4 (E) 2

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39. A pinochle deck consists of 48 cards, 8 of which are aces. A standard deck consists of 52 cards, 4 of which are aces. You have one of each type of deck, but cannot remember which is which. You pick up one of the decks and look at the top card. It is an ace. The probability that you picked up the pinochle deck is:

(A) 1/2 (B) 2/3 (C) 6/13 (D) 13/19 (E) none of these

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40. The solution set to the equation

$$4^{x+1}-2^x=2^{x+2}-1$$

is:

(A) $\emptyset$ (B) $(-\infty,\infty)$ (C) $\{0\}$ (D) $\{-1\}$ (E) none of these

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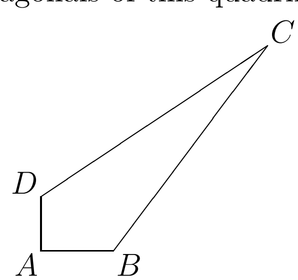
41. In the figure, suppose $AB=4$, $BC=12$, $CD=13$, and $DA=3$. If $\angle BAD$ is a right angle, then the product of the lengths of the diagonals of this quadrilateral is:

(A) 63.75
(B) $8\sqrt{85}$
(C) $10\sqrt{74}$
(D) $4\sqrt{1780}$
(E) none of these



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42. The value of $\tan(\pi/24)$ is:

(A) $\sqrt{\dfrac{2-\sqrt{2+\sqrt{3}}}{2+\sqrt{2+\sqrt{3}}}}$ (B) $\dfrac{\sqrt{2-\sqrt{3}}}{2}$ (C) $\dfrac{\sqrt{2+\sqrt{3}}}{2}$ (D) $\sqrt{7-4\sqrt{3}}$

(E) $\dfrac{\sqrt{3}}{3}$

---

43. The sum of the reciprocals of the zeros of x^3+3x^2-7x+2 is:

(A) 3.5 (B) 1/3 (C) $-1/3$ (D) -3.5 (E) none of these

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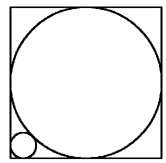
44. For real numbers a and b, define $a * b = a + b + ab$. Suppose that e is a real number such that $x * e = x$ for all x. A real number y for which there does not exist a real number z with $y * z = e$ is:

(A) e (B) $-e$ (C) 1 (D) -1 (E) none of these

---

45. A circle of radius 1 cm is inscribed in a square. A second circle is drawn externally tangent to this circle and tangent to two sides of the square as shown in the figure. The radius of this second circle, in cm, is:

(A) $1 - \sqrt{2}/2$
(B) $3 - 2\sqrt{2}$
(C) $\sqrt{2}/8$
(D) $1/4$
(E) none of these



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46. An interval that contains a value of x for which it is possible to find a, b, y, and z to solve the system

$$\begin{aligned} 2ax + b &= -1 \\ 2ay + b &= 2 \\ 2az + b &= -1 \\ x^2 + y^2 + z^2 &= 1 \\ x + y + z &= 0 \end{aligned}$$

is:

(A) $[-1, -1/2]$ (B) $[0, 1/2]$ (C) $[1/2, 1]$ (D) $[1, 2]$ (E) none of these

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47. Consider the polynomial $f(x) = x^2 - 2$. If t is a real number such that $(f \circ f)(t) = 0$, then the multiplicative inverse of t is:

(A) $0.5\sqrt{2+\sqrt{2}}$ (B) $-0.5\sqrt{2+\sqrt{2}}$ (C) $0.5t^3 - 2t$ (D) $2t - 0.5t^3$
(E) none of these

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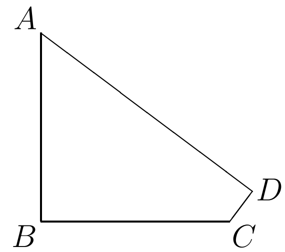
48. In the figure, suppose $AB = BC = 5$, $CD = 1$, and $AD = 7$. If $\angle ABC$ is a right angle, then $\cos \angle BAD$ is:



(A) 3/5
(B) 4/5
(C) 5/7
(D) 7/25
(E) none of these

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49. Given $\dfrac{x-y+1}{x-y-1} = a$ and $\dfrac{x+y+1}{x+y-1} = b$, the value for y is:

(A) $\dfrac{a+b}{(a-1)(b-1)}$ (B) $\dfrac{a+b}{(a+1)(b+1)}$ (C) $\dfrac{ab-1}{(a-1)(b-1)}$

(D) $\dfrac{ab}{(a-1)(b-1)}$ (E) $\dfrac{a-b}{(a-1)(b-1)}$

---

50. Two sides of triangle A have lengths 15 and 25, respectively. Two sides of triangle B have lengths 30 and 36, respectively. If A and B are similar acute triangles, the geometric mean of their unknown sides is:

(A) $6\sqrt{10}$ (B) $15\sqrt{2}$ (C) $9\sqrt{5}$ (D) $\sqrt{30}$ (E) 30

Answer Key

1. B
2. C
3. B
4. C
5. A
6. E
7. A
8. C
9. E
10. D
11. D
12. B
13. D
14. A
15. D
16. C
17. C
18. A
19. C
20. C
21. E
22. B
23. A
24. E
25. B
26. D
27. C
28. A
29. D
30. D
31. E
32. C
33. A
34. E
35. C
36. B
37. E
38. E
39. D
40. E
41. B
42. A
43. A
44. D
45. B
46. B
47. D
48. A
49. E
50. B