Mathematics Competition

Indiana University of Pennsylvania

2009

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. The mass of a liquid varies directly as its volume. If the mass of the liquid in a cubical container 5 cm on a side is 375 g, then the mass of the liquid in a cubical container 4 cm on a side is:

(A) 192 g (B) 144 g (C) 64 g (D) 12 g (E) none of these

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2. A cross country skier going at a constant speed reaches the 12 km mark of a race 40 minutes after reaching the 2 km mark. The speed of the skier in km/hr is:

(A) 6 (B) 10 (C) 15 (D) 18 (E) 20

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3. The number of positive integers less than or equal to 15,000 that are divisible by both 48 and 72 is:

(A) 104 (B) 125 (C) 208 (D) 625 (E) none of these

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4. The solution set to the inequality

$$2x + 2(x - 10) > 4x - 12$$

is:

(A) $\emptyset$ (B) $(-\infty, \infty)$ (C) $(-\infty, 0)$ (D) $(0, \infty)$ (E) none of these

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5. If $\tan\theta = -3/2$ and $\cos\theta < 0$ then $\csc\theta$ is equal to:

(A) $\dfrac{2\sqrt{13}}{13}$ (B) $\dfrac{3\sqrt{13}}{13}$ (C) $\dfrac{-3\sqrt{13}}{13}$ (D) $\dfrac{\sqrt{13}}{3}$ (E) none of these

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6. Company A rents copiers for a monthly charge of \$240 plus 12 cents per copy. Company B rents copiers for a monthly charge of \$480 plus 6 cents per copy. Company A is the more expensive option if and only if the number of copies per month is greater than:

(A) 40 (B) 60 (C) 4000 (D) 6000 (E) none of these

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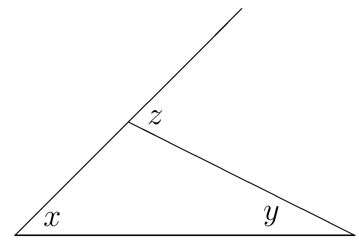
7. Suppose there is some n such that in the accompanying figure $m\angle x = (4n - 19)^\circ$, $m\angle y = (n + 9)^\circ$, and $m\angle z = (166 - 6n)^\circ$. Then the measure of the one of these that is an exterior angle of the triangle is:

(A) 70°
(B) 84°
(C) 135°
(D) 160°
(E) none of these



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8. The number of ounces of a $12\frac{1}{2}\%$ acid solution that must be added to 10 ounces of a 20% acid solution to obtain a $16\frac{2}{3}\%$ acid solution is:

(A) 6 (B) 8 (C) 10 (D) 12 (E) none of these

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9. If the function f is defined by

$$f(x) = \sqrt[4]{\frac{2-x}{x}},$$

then the domain of f is:

(A) $(-\infty, 0) \cup [2, \infty)$ (B) $(-\infty, 0) \cup (0, 2]$ (C) $[2, \infty)$ (D) $[0, 2)$
(E) $(0, 2]$

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10. If $\log_3 5^x = \log_7 5$, then x is equal to:

(A) $\log_3 5$ (B) $\log_5 3$ (C) $\log_7 3$ (D) $\log_7 5$ (E) none of these

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11. Jill starts 11 km away from Joe. Both begin to walk toward each other at the same time. Jill walks 2.5 km/hr. They meet in 2 hours. The speed in km/hr at which Joe is walking is:

(A) 3 (B) 4 (C) 5 (D) 2.5 (E) none of these

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12. The number of distinct 5-digit numbers that can be formed from the digits 1, 1, 1, 3, 5 is:

(A) 6 (B) 8 (C) 10 (D) 15 (E) none of these

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13. The solution set to the equation

$$|2^x - 5| = 3$$

is:

(A) $\emptyset$ (B) $\{-1\}$ (C) $\{1\}$ (D) $\{3\}$ (E) none of these

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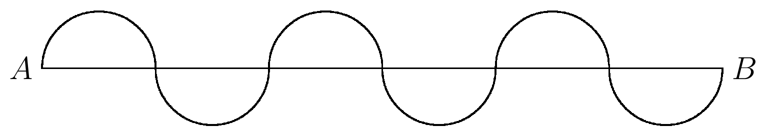
14. Assume that all of the arcs in the accompanying figure are semicircles with the same radius. The ratio of the length of the meandering path from A to B to the length of the straight path from A to B is:

(A) 1
(B) $\pi/4$
(C) $\pi/2$
(D) π
(E) 2π



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15. Suppose x, y, and z are nonzero numbers for which

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$

The only false statement among the following is:

(A) $yz + xz = xy$ (B) $\dfrac{y+x}{xy} = \dfrac{1}{z}$ (C) $z = \dfrac{xy}{x+y}$ (D) $x = \dfrac{yz}{y-z}$
(E) $y = \dfrac{xz}{x+z}$

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16. For $|x| \leq 1$, one can show that $\arcsin x + \arccos x$ is equal to:

(A) $\arctan x$ (B) $\dfrac{\cos x + \sin x}{\sin x \cos x}$ (C) π (D) $\pi/2$ (E) none of these

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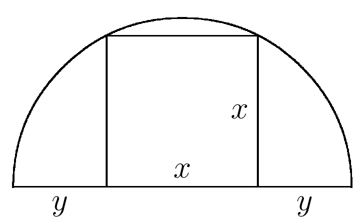
17. The figure shows a square inscribed in a semicircle. Given that the length of y is 6, the length of x is:

(A) 9
(B) $6 - 2\sqrt{3}$
(C) $6 + 2\sqrt{3}$
(D) $3 - 3\sqrt{5}$
(E) $3 + 3\sqrt{5}$



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18. A collection of 19th century coins containing only 2-cent, 3-cent, and 20-cent pieces has a total face value of \$1.38. If the number of 3-cent pieces is twice the number of 20-cent pieces and the number of 2-cent pieces is one less than the number of 20-cent pieces, then the number of 2-cent pieces is:

(A) 10 (B) 8 (C) 6 (D) 5 (E) 4

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19. Suppose the sum of the first $3n$ positive integers is 150 more than the sum of the first n positive integers. Then the sum of the first $4n$ positive integers is:

(A) 136 (B) 210 (C) 300 (D) 406 (E) 528

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20. Suppose m is a positive real number. Then the roots of

$$2mx^2 + (5m + 2)x + 4m + 1 = 0$$

are not real if and only if:

(A) $m > 1$ (B) $m > 2$ (C) $m > 3$ (D) $m > 4$ (E) $m > 5$

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21. The solution (x, y) to the system of equations

$$\begin{aligned} 8^y &= 4^{2x+3} \\ \log_2 y &= 4 + \log_2 x \end{aligned}$$

is:

(A) $(3/22, 11/24)$ (B) $(3/22, 24/11)$ (C) $(1, 2)$ (D) $(-3/2, 0)$
(E) none of these

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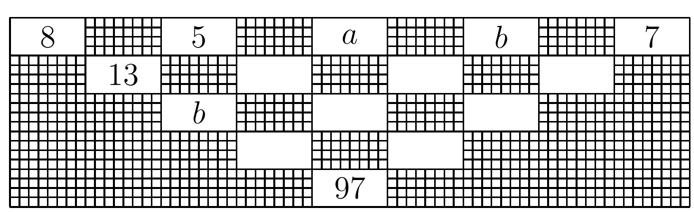
22. The number in each unshaded space is obtained by adding the numbers connected to it diagonally from the row above. The value of a is:

(A) -2
(B) -1
(C) 1
(D) 2
(E) none of these



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23. An interval that contains the x values of all pairs (x, y) that satisfy the system

$$\begin{aligned} x^2 - y^2 &= 1 \\ x^3 - y^2 &= x \end{aligned}$$

is:

(A) $[-6, -2)$ (B) $[-2, 2)$ (C) $[2, 6)$ (D) $[6, 10)$ (E) $[10, \infty)$

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24. The number of solutions to the equation

$$4 \sin^2 x \cos^2 x - 8 \sin^2 x + 3 \cos^2 x - 6 = 0$$

in the interval $[0, \pi]$ is:

(A) 0 (B) 2 (C) 4 (D) 8 (E) none of these

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25. John owns a hot dog stand. He found that his daily profit in dollars is represented by

$$p = -x^2 + 50x - 67$$

where x is the number of hot dogs sold. The maximum daily profit John can earn is:

(A) \$25 (B) \$67 (C) \$327 (D) \$558 (E) \$692

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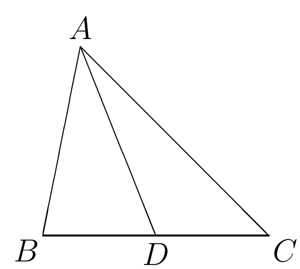
26. Suppose in the figure $AB = 5$, $AC = 7$, and $BD = 3 = DC$. The length of $\overline{AD}$ is:

(A) 4
(B) $\sqrt{17}$
(C) $3\sqrt{2}$
(D) $2\sqrt{7}$
(E) $\sqrt{19}$



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27. The solution set to the equation

$$|\ln x| + 3 \ln |x| = 8$$

is:

(A) $\{e^2\}$ (B) $\{e^2, e^4\}$ (C) $\{\pm e^2\}$ (D) $\{\pm e^2, \pm e^4\}$ (E) none of these

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28. Assuming b is an integer, the number of quadratic polynomials of the form $3x^2+bx-2$ that can be factored using integer coefficients is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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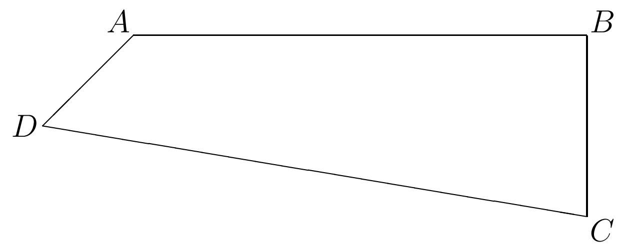
29. The quadrilateral $ABCD$ has side lengths $AB=12$, $BC=5$, $CD=15$, and $AD=4$. If $\angle ABC$ is a right angle, then the area of the quadrilateral is:

(A) 48
(B) 54
(C) 60
(D) 72
(E) none of these



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30. Suppose it takes Alex 10 hours to complete a job and Chris 8 hours to complete the same job. If Alex and Bob take 8 hours to do the job together, then the number of hours it would take Bob and Chris together is:

(A) $4\frac{2}{3}$ (B) $5\frac{3}{5}$ (C) $6\frac{1}{4}$ (D) $6\frac{2}{3}$ (E) $7\frac{2}{5}$

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31. A six-sided prime die has the numbers 2, 3, 5, 7, 11, and 13 on the faces. Suppose that three six-sided prime dice are rolled. The probability that the sum of the three faces showing on the top of the dice is greater than 25 is equal to:

(A) $\dfrac{53}{216}$ (B) $\dfrac{59}{216}$ (C) $\dfrac{61}{216}$ (D) $\dfrac{67}{216}$ (E) $\dfrac{71}{216}$

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32. The solution set, in interval notation, to the inequality

$$\frac{e^x \ln x}{\sin x} \geq 0$$

in the interval $(0, 2\pi)$ is:

(A) $(0,1]$ (B) $[1,\pi)$ (C) $(\pi,2\pi)$ (D) $[1,2\pi)$ (E) $(0,2\pi)$

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33. The set of all x-coordinates of pairs (x, y) that satisfy the system

$$\begin{aligned} x^2 + x + y^2 - 3y + 2 &= 0 \\ x + 1 + \frac{y^2 - y}{x} &= 0 \end{aligned}$$

is:

(A) $\{-1\}$ (B) $\{-1, 0\}$ (C) $\{-1, 1\}$ (D) $\{1\}$ (E) none of these

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34. A circle of radius 1 and a circle of radius 2 are externally tangent. Two distinct lines are drawn from the center of the larger circle, each tangent to the smaller circle. The distance between the points where these lines touch the smaller circle is:

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{2\sqrt{2}}{3}$ (C) $\frac{4\sqrt{2}}{3}$ (D) $\sqrt{3}$ (E) none of these

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35. If a, b, and c are all nonzero, the solution set to the equation

$$ax\left(\frac{ax}{b^2} - \frac{1}{c}\right) + \frac{1}{c}\left(\frac{b^2}{c} - ax\right) = 0$$

is:

(A) $\left\{\frac{b^2}{ac}\right\}$ (B) $\left\{\frac{2ab^2c \pm \sqrt{a^2 - 4b^2c^2}}{2a^2}\right\}$ (C) $\left\{\frac{bc^2 \pm \sqrt{bc^2}}{2b}\right\}$

(D) $\left\{\frac{c^2}{2a^2}\right\}$ (E) none of these

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36. The expression

$$\sqrt{2} \cdot \sqrt[4]{2} \cdot \sqrt[8]{2} \cdot \sqrt[16]{2} \cdot \sqrt[32]{2} \cdot \ldots$$

is equal to:

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

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37. The hypotenuse of a right triangle is 8 feet less than three times the shorter leg and the longer leg is 8 feet more than twice the shorter leg. The area of the triangle in sq. ft. is:

(A) 270 (B) 396 (C) 480 (D) 1070 (E) none of these

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38. One subset of the solution set to the inequality

$$\frac{x}{x-2} \geq \frac{3}{x+3} + \frac{7}{x^2+x-6}$$

is:

(A) $(-\infty, -3]$ (B) $(-3, -1]$ (C) $(-1, 1)$ (D) $[1, \infty)$
(E) none of these

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39. The solution set to the equation

$$\log_3(2-3x) = \log_9(6x^2 - 19x + 2)$$

is:

(A) $\{-1/3, 1/3\}$ (B) $\{0, -1/3\}$ (C) $\{1/3, 1\}$ (D) $\{-1/3, -2\}$
(E) none of these

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40. The number of ordered pair solutions (x, y) with real coordinates of the system

$$\begin{aligned} x^2 + y^2 + 2y &= 8 \\ 4x^2 + 9y^2 &= 36 \end{aligned}$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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41. Cannonballs may be stacked in triangular pyramids: four cannonballs using two levels (by placing three in triangular form and one on top), ten cannonballs using three levels, etc. John has 200 cannonballs on his ship. The number of cannonballs that can be added to complete the smallest pyramid containing all 200 cannonballs is:

(A) 10 (B) 25 (C) 56 (D) 89 (E) none of these

---

42. A diagonal is drawn in a 3 by 4 rectangle. Circles are inscribed in each of the two triangles thus formed. The distance between the centers of these circles is:

(A) 1 (B) $\sqrt{3}$ (C) 2 (D) $\sqrt{5}$ (E) none of these

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43. If $a > b > 0$, then the solution set to the equation

$$\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{x+b}+\sqrt{x-b}} = \frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{x+b}-\sqrt{x-b}}$$

includes

(A) $a^2 - b^2$ (B) $b^2 - a^2$ (C) $\sqrt{ab}$ (D) $\dfrac{a+b}{2}$ (E) $\dfrac{a-b}{2}$

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44. The number of pairs of prime numbers p and q not exceeding 2311 for which $p-q = 131$ is equal to:

(A) 1 (B) 3 (C) 8 (D) 17 (E) none of these

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45. The solution set to the equation

$$\log_7 x - 2\log_x 7 = 1$$

is:

(A) $\{1, 7\}$ (B) $\{7, 49\}$ (C) $\{1/7\}$ (D) $\{1/7, 49\}$ (E) none of these

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46. Let r and s be roots of the equation

$$9x^2 - 2x - 2 = x\sqrt{5}(4x - 1)$$

with $r < s$. Then $s - r$ is equal to:

(A) $\dfrac{\sqrt{5}-2}{3}$ (B) $\dfrac{\sqrt{5}+2}{3}$ (C) $3\sqrt{5}+6$ (D) $3\sqrt{5}-6$

(E) none of these

---

47. A small sack contains 3 red marbles, 3 blue marbles, and 6 white marbles. If three marbles are drawn without replacement from the sack, then the probability of choosing at least one red marble and at least one blue marble is:

(A) $\dfrac{27}{110}$ (B) $\dfrac{18}{55}$ (C) $\left(\dfrac{34}{55}\right)^2$ (D) $\dfrac{9}{22}$ (E) $\dfrac{10}{11}$

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48. An airplane starts at the equator and flies continuously northeast (45 degrees north of east). Assuming the earth is a perfect sphere of radius R, the distance the plane flies until it reaches the north pole is:

(A) $\sqrt{2}\pi R$ (B) $\sqrt{2}\pi R/2$ (C) $\pi R/2$ (D) $2\sqrt{3}\pi R$ (E) infinite

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49. The number of distinct real solutions to the equation

$$x^4 - 8x^3 + 17x^2 + 6x = 36$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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50. The expression $4 \sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ \sin 90^\circ$ is equal to:

(A) $\sin 70^\circ$ (B) 4 (C) 2 (D) 1 (E) 1/4

Answer Key

1. A
2. C
3. A
4. A
5. D
6. C
7. A
8. B
9. E
10. C
11. A
12. E
13. E
14. C
15. E
16. D
17. E
18. E
19. C
20. B
21. B
22. B
23. B
24. A
25. D
26. D
27. A
28. E
29. B
30. D
31. A
32. B
33. A
34. C
35. A
36. B
37. C
38. C
39. D
40. D
41. E
42. D
43. C
44. E
45. D
46. C
47. B
48. B
49. D
50. E