Mathematics Competition

Indiana University of Pennsylvania

2011

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.

2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.

3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.

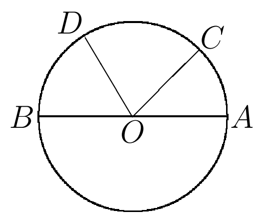
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.

5. Use of pencil, eraser, and scratch paper only are permitted.

6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. On circle O, points C and D are on the same side of diameter $\overline{AB}$, $\angle AOC = 60°$, and $\angle DOB = 45°$. The ratio of the area of the smaller sector COD to the area of the circle is:



(A) 1/4
(B) 2/9
(C) 3/10
(D) 5/18
(E) 5/24

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2. If $2x + 3y = 7$ and $3x + 4y = 12$, then $x + y$ is equal to:

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

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3. A basketball player made five baskets during a game. Each basket was worth either two or three points. The number of distinct scores that this player could have reached is:

(A) 2 (B) 3 (C) 4 (D) 6 (E) none of these

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4. The sum of the squares of the digits of a positive two-digit number is 61. The tens digit is 1 more than the units digit. The number is:

(A) 54 (B) 45 (C) 65 (D) 56 (E) 76

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5. If $\tan\theta = -\sqrt{3}$, and $-\pi/2 < \theta < \pi/2$, then the value of $\cos\theta$ is:

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{-1}{2}$ (D) $\frac{-\sqrt{3}}{2}$ (E) $-\sqrt{3}$

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6. The hypotenuse of a right triangle is 8 feet less than three times the shorter leg, and the longer leg is 8 feet more than twice the shorter leg. The lengths of the three sides of the triangle are:

(A) 18 ft, 44 ft, 46 ft (B) 15 ft, 36 ft, 39 ft (C) 30 ft, 68 ft, 82 ft
(D) 20 ft, 48 ft, 52 ft (E) none of these

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7. Construct a new 4×4 array of numbers from the one given by reversing the order of the numbers in the second and fourth rows. The absolute value of the difference of the two diagonal sums in the resulting array is:

1	2	3	4
8	9	10	11
15	16	17	18
22	23	24	25

(A) 2
(B) 4
(C) 6
(D) 8
(E) none of these

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8. The solution set to the system of equations

$$\begin{aligned} x^2 &= 6x - y \\ y &= 3x + 2 \end{aligned}$$

is:

(A) $\{1,2)\}$ (B) $\{5,8)\}$ (C) $\{(-1,5),(-2,8)\}$
(D) $\{(-1,-1),(-2,-4)\}$ (E) $\{(1,5),(2,8)\}$

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9. Side lengths of a triangle are 7, 20, and 23. The shape of the triangle is:

(A) acute (B) right (C) obtuse (D) impossible to determine
(E) none of these

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10. If $\log_a 512 = c$, then the percentage of the value of c that $\log_a 8$ represents is:

(A) 20% (B) $33\frac{1}{3}\%$ (C) 50% (D) $66\frac{2}{3}\%$ (E) 8%

---

11. The solution set, in interval notation, of the inequality

$$2x - 8 < 4 < 2x + 4$$

is:

(A) $(-2,1)$ (B) $(0,6)$ (C) $(-12,0)$ (D) $(-3,0)$ (E) $(-\infty,\infty)$

---

12. The probability that Alexis practices the piano is 0.85 if it rains and 0.32 if it does not rain. There is a 25% chance of rain. The probability that Alexis practices the piano is:

(A) 0.75 (B) 0.21 (C) 0.24 (D) 0.45 (E) none of these

---

13. For real numbers A and B, define $A\$B = (A-B)^2$. The expression $(X-Y)^2\$(Y-X)^2$ is equal to:

(A) 0 (B) X^2+Y^2 (C) $2X^2$ (D) $2Y^2$ (E) $4XY$

---

14. A cone-shaped mountain has its base on the ocean floor and has a height of 12,000 ft. The top 1/8 of the volume of the mountain is above water. The depth of the ocean at the base of the mountain is:

(A) 6000 ft (B) $3000(4-\sqrt{2})$ ft (C) 4000 ft (D) 7000 ft
(E) none of these

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15. The distance from the origin to the point of intersection of the lines $y = 3x - 5$ and $y = -4x + 16$ is:

(A) 3 (B) 5 (C) $2\sqrt{2}$ (D) $\sqrt{3}$ (E) none of these

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16. If $0 \le \theta \le \pi$, then the solution set of the equation

$$\sin 2\theta = \cos\theta$$

is:

(A) $\left\{0, \frac{\pi}{2}, \pi\right\}$ (B) $\left\{\frac{\pi}{6}, \frac{\pi}{2}\right\}$ (C) $\left\{\frac{\pi}{3}, \frac{4\pi}{3}\right\}$ (D) $\left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{4\pi}{3}\right\}$
(E) $\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\right\}$

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17. High quality golf balls cost \$3.00 each and low quality golf balls cost \$2.00 each. The number of different buckets of golf balls that can be purchased for exactly \$50 is:

(A) 1 (B) 8 (C) 16 (D) 17 (E) none of these

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18. The smallest value of x in the solution pairs of the system

$$\begin{aligned} x^2 + xy &= 40 \\ xy + y^2 &= 24 \end{aligned}$$

is:

(A) -20 (B) -8 (C) -5 (D) -2 (E) -1

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19. The solution set of the equation

$$\log_2 x + \log_2(x-1) = 1$$

is:

(A) $\{1\}$ (B) $\{-1\}$ (C) $\{2\}$ (D) $\{-1, 2\}$ (E) $\{1, -2\}$

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20. An organization has 1000 members. Two of the members, Smith and Jones, are running for president of the organization. If 60% of the males and 35% of the females vote for Smith, giving Smith 510 votes, then the number of members who are male is:

(A) 680 (B) 640 (C) 600 (D) 560 (E) 500

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21. Points A and B are on a circle of radius 13 and $AB = 10$. Point C is the midpoint of the minor arc AB. The length of the line segment $\overline{AC}$ is:

(A) $\sqrt{26}$ (B) 16/3 (C) $\sqrt{30}$ (D) 6 (E) $\sqrt{39}$

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22. A correct simplification of the expression

$$\tan(\arcsin x)$$

is:

(A) $\dfrac{x}{\sqrt{1-x^2}}$ (B) $\dfrac{x}{\sqrt{1+x^2}}$ (C) $\dfrac{x}{\sqrt{x^2-1}}$ (D) $\dfrac{-x}{\sqrt{1-x^2}}$

(E) $\dfrac{-x}{\sqrt{1+x^2}}$

---

23. The number of distinct real solutions to the equation

$$|x^2 - 6x + 1| = 8$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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24. Points B and C lie on $\overline{AD}$. The length of $\overline{AB}$ is four times the length of $\overline{BD}$, and the length of $\overline{AC}$ is nine times the length of $\overline{CD}$. The ratio of the length of $\overline{BC}$ to the length of $\overline{AD}$ is:

(A) $\dfrac{1}{36}$ (B) $\dfrac{1}{13}$ (C) $\dfrac{1}{10}$ (D) $\dfrac{1}{5}$ (E) none of these

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25. A teacher allows her students to work on an exam for up to 60 minutes. They may leave at any time, but cannot return. No students leave during the first 15 minutes, but after that the number of students remaining follows precisely an exponential decay rate. Noting that 180 students stay at least half the time and 150 stay at least 3/4 of the time, the number of students who did not stay the entire time is:

(A) 89 (B) 91 (C) 93 (D) 97 (E) 101

---

26. Suppose E is on segment $\overline{PQ}$ and $PE = \frac{1}{4}PQ$. Let $P = (-3, 3)$ and $Q = (5, 7)$. Then the distance from point E to point $F = (1, -1)$ is:

(A) $\sqrt{53}$ (B) 6 (C) $\sqrt{47}$ (D) $2\sqrt{10}$ (E) $\sqrt{29}$

---

27. The number of distinguishable ways that the letters of the word GEOLOGY can be arranged is:

(A) 630 (B) 1260 (C) 2520 (D) 5040 (E) none of these

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28. If $x^3 - 2x^2 + ax + b$ is divisible by both $x - 2$ and $x - 4$, then the value of a is:

(A) 64 (B) -32 (C) 6 (D) -16 (E) none of these

---

29. A circle has a radius of $\log_7 a$ and a circumference of $\log_7 b^4$. The value of $\log_a b$ is:

(A) 7^π (B) 14π (C) $\frac{1}{4\pi}$ (D) $\frac{2}{\pi}$ (E) $\frac{\pi}{2}$

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30. Two workers, Joe and Joanne, need to fill an order. Working alone Joe could produce the items in the order in 10 hours. Joanne could produce them in 9 hours. Working together, they tend to talk and their combined output drops by 10 items per hour. If together they take 5 hours to fill the order, the total number of items in the order is:

(A) 500 (B) 900 (C) 950 (D) 1000 (E) 1200

---

31. The distance between the lines $y = 2x + 1$ and $y = 2x + 11$ is:

(A) 10 (B) 5 (C) $2\sqrt{5}$ (D) $3\sqrt{2}$ (E) none of these

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32. Consider these statements.

$$\begin{array}{ll} \text{I.} & \cos(\cos^{-1}(-\sqrt{2}/2)) = -\sqrt{2}/2 \\ \text{II.} & \cos^{-1}(\cos(-\sqrt{2}/2)) = -\sqrt{2}/2 \\ \text{III.} & \sin(\cos^{-1}(-\sqrt{2}/2)) = -\sqrt{2}/2 \end{array}$$

The set of true statements here is:

(A) {I} (B) {II} (C) {I, II} (D) {I, III} (E) {I, II, III}

---

33. An interval that does <u>not</u> contain a solution to

$$\sqrt{x-3} - \sqrt{x-4} - \sqrt{4x-15} = 0$$

is:

(A) $(-3, 4)$ (B) $(2, 5)$ (C) $(3, 7)$ (D) $(1, 6)$ (E) none of these

---

34. A ladder is leaning against a wall. It slips down with its top always remaining in contact with the wall and its bottom always in contact with the floor. The center point traces out the path of:

(A) a straight line segment
(B) an arc of the sine function
(C) an arc of a parabola
(D) an arc of a hyperbola
(E) an arc of a circle

---

35. In golf, par is the number of strokes that a golfer is expected to take to complete play on a hole. On an 18 hole golf course, there are par-3, par-4, and par-5 holes. A golfer who shoots par on every hole takes 73 strokes. If the sum of the number of par-3 holes and the number of par-5 holes is 13, then the number of each type of hole is:

(A) 5 par-3, 5 par-4, 8 par-5
(B) 8 par-3, 5 par-4, 5 par-5
(C) 7 par-3, 5 par-4, 6 par-5
(D) 6 par-3, 5 par-4, 7 par-5
(E) none of these

---

36. For each positive integer n, the mean of the first n terms of a sequence is n. The 893rd term of the sequence is:

(A) 893 (B) 893^2 (C) 1786 (D) 1785 (E) none of these

---

37. Vertex E of equilateral $\triangle ABE$ is in the interior of unit square $ABCD$. Let R be the region consisting of all points inside $ABCD$ and outside $\triangle ABE$ whose distance from $\overline{AD}$ is between $1/4$ and $3/4$. The area of region R is:

(A) $\dfrac{\sqrt{3}}{8}$ (B) $\dfrac{2-\sqrt{3}}{5}$ (C) $\dfrac{4+\sqrt{3}}{16}$ (D) $\dfrac{8-3\sqrt{3}}{32}$ (E) $\dfrac{8-3\sqrt{3}}{16}$

---

38. For a solution to the system

$$\begin{aligned} 6w + 2x - 6y + z &= -18 \\ -w + x + 3y - z &= 4 \\ 2w - x + y + 3z &= 6 \\ w + x + 6y - z &= 17 \end{aligned}$$

the value of $w + x + y + z$ is:

(A) -8 (B) -15 (C) -2 (D) 0 (E) none of these

---

39. The solution set to the equation $4^{x-x^2} = \frac{1}{2}$ is:

(A) $\left\{\dfrac{\ln 2}{\ln 4}\right\}$ (B) $\left\{\dfrac{-\ln 2}{\ln 4}\right\}$ (C) $\{\pm \log_4 2\}$ (D) $\left\{\pm\dfrac{\sqrt{3}}{2}\right\}$
(E) $\left\{\dfrac{1\pm\sqrt{3}}{2}\right\}$

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40. The sum of the squares of the roots of the polynomial $x^3 + 4x^2 - 5x + 17$ are:

(A) 26 (B) 17 (C) 16 (D) -5 (E) none of these

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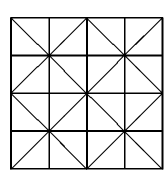
41. The number of triangles in the accompanying figure is:



(A) 32
(B) 76
(C) 84
(D) 96
(E) none of these

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42. Let x and y be natural numbers with $x < y$. In a rectangular room measuring x feet by y feet, 1×1 foot square tiles are laid against the walls. The total area of the tiled region is one-half the area of the entire room. The number of possible such ordered pairs (x, y) is:

(A) 5 (B) 4 (C) 3 (D) 2 (E) none of these

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43. If $x^4 + ax^3 + bx^2 + cx + d$ has $\sqrt{2} + \sqrt{5}$ as a root and a, b, c, and d are rational, then the value of b is:

(A) -14 (B) -6 (C) 3 (D) 6 (E) 14

---

44. Let A, B, and C be three distinct points on the graph of $y = x^2$ such that $\overline{AB}$ is parallel to the x-axis and $\triangle ABC$ is a right triangle with area 2011. The sum of the digits of the y-coordinate of C is:

(A) 13 (B) 14 (C) 15 (D) 16 (E) 17

---

45. The sum of the base 10 logarithms of the divisors of 10^n is 450. In this case, n is:

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

---

46. A positive integer whose first digit is 1 is tripled when you move that first 1 to the end of the number. The number of digits in the smallest such integer is:

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

---

47. A pyramid has a square base $ABCD$ and vertex E. The area of square $ABCD$ is 289, and the areas of $\triangle ABE$ and $\triangle CDE$ are $212\frac{1}{2}$ and 221, respectively. The volume of the pyramid is:

(A) 1156 (B) $578\sqrt{6}$ (C) $1156\sqrt{2}$ (D) $1156\sqrt{3}$ (E) 2312

---

48. Let $ABCD$ be a trapezoid with $AB \parallel CD$, $AB = 9$, $BC = 5$, $CD = 17$, and $DA = 7$. Bisectors of $\angle A$ and $\angle D$ meet at P, and bisectors of $\angle C$ and $\angle D$ meet at Q. The area of $ABQCDP$ is:

(A) $24\sqrt{3}$ (B) $25\sqrt{3}$ (C) $27\sqrt{3}$ (D) $28\sqrt{3}$ (E) $30\sqrt{3}$

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49. Let a and b be positive integers with $b > a$. The absolute value of the difference of the values of x that solve

$$\frac{1}{2x^2+x-1} + \frac{1}{2x^2-3x+1} = \frac{a}{2bx-b} - \frac{2bx+b}{ax^2-a}$$

is:

(A) $\dfrac{4b^2-2a^2}{4b^2-a^2}$ (B) $\dfrac{b-a}{4b^2-a^2}$ (C) $\dfrac{2ab}{4b^2-a^2}$ (D) $\dfrac{b+a}{4b^2-a^2}$ (E) $\dfrac{b+a}{2b+a}$

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50. On a long table there is a row of 15 empty cake boxes. Carlos, the Cake Boss, randomly places cakes into 12 of the boxes. The probability that two consecutive boxes on the table remain empty is:

(A) 1/3 (B) 12/35 (C) 13/35 (D) 2/5 (E) none of these

Answer Key

1. E
2. C
3. D
4. C
5. A
6. D
7. B
8. E
9. C
10. B
11. B
12. D
13. A
14. A
15. B
16. E
17. E
18. C
19. C
20. B
21. A
22. A
23. D
24. C
25. B
26. E
27. B
28. D
29. E
30. B
31. C
32. A
33. A
34. E
35. D
36. D
37. E
38. C
39. E
40. A
41. D
42. D
43. A
44. C
45. B
46. D
47. E
48. B
49. A
50. C