Mathematics Competition

Indiana University of Pennsylvania

1996

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.

2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.

3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.

4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.

5. Use of pencil, eraser, and scratch paper only are permitted.

6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. Define an operation $\odot$ on the real numbers by $x \odot y = x^2 - xy$. The condition that is necessary to ensure that $x \odot y = y \odot x$ is:

   (A) $x = 0$ (B) $x = y$ (C) $y = 0$ (D) $x = -y$ (E) $x^2 = y^2$

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2. In simplified form $\dfrac{2x^2 + x - 10}{3x^2 + 7x + 4} \div \dfrac{x^2 - 3x + 2}{3x^2 + x - 4}$ is:

   (A) 1 (B) $\dfrac{2x+5}{x+1}$ (C) $\dfrac{2x+5}{x-1}$ (D) $\dfrac{3x+4}{x-2}$ (E) $\dfrac{x-1}{x+1}$

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3. A square has twice the perimeter of a circle. The ratio of the area of the square to the area of the circle is:

   (A) 4 (B) π (C) $1/\pi$ (D) $16/\pi$ (E) none of these

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4. The solution set of the equation

$$\frac{x-3}{3} - \frac{x-19}{4} = \frac{1-2x}{2}$$

   is:

   (A) $\{\ \}$ (B) $\{0, -3\}$ (C) $\{-3\}$ (D) $\{1\}$ (E) none of these

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5. If $x = \log_b 2$ and $y = \log_b 5$, then $\log_b \sqrt{12.5}$ is equal to:

   (A) $(y^2 - x)/2$ (B) $(x^2 - y)/2$ (C) $(2y - x)^{1/2}$ (D) $y - x/2$
   (E) none of these

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6. A man leaves his home and drives to the beach at an average speed of 50 mph and returns home at an average speed of 30 mph. If the round trip takes 2 hours, the distance from home to the beach is:

   (A) 100 miles (B) 40 miles (C) 37.5 miles (D) $33.\bar{3}$ miles
   (E) 10 miles

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7. If today were Tuesday, then 5387 days from now it would be:

   (A) Tuesday (B) Thursday (C) Friday (D) Sunday
   (E) none of these

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8. If in the expression x^2y the values of x and y are each decreased by 20%, then the value of the expression is decreased by a factor of:

(A) 61/125 (B) 50% (C) 64/125 (D) 60% (E) none of these

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9. A 10 ft tall giraffe walks around the earth. The distance his head travels exceeds the distance his feet travel by:

(A) 10π ft (B) 1024 ft (C) 20π ft (D) 20 ft (E) none of these

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10. For every positive real number x,

$$\tan^{-1}(x) + \tan^{-1}(1/x)$$

is:

(A) 1 (B) 0 (C) $\pi/2$ (D) x (E) -1

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11. On the following inequalities, the one that is not equivalent to the others is:

(A) $|-2x-7| < 7$ (B) $-7 < -2x-7 < 7$ (C) $0 < -2x < 14$
(D) $0 < x < -7$ (E) $-7 < x < 0$

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12. The expression $\dfrac{\log_2 4 \log_4 3}{\log_3 5 \log_2 3}$ is equal to:

(A) 0 (B) 1 (C) $\log_5 3$ (D) $\ln 2$ (E) e

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13. If the ratio of $3x-7y+4$ to $4x+3y+12$ is $1/3$, then the ratio of x to y is:

(A) $\dfrac{-18}{5}$ (B) $\dfrac{5}{18}$ (C) 2 (D) $\dfrac{24}{5}$ (E) none of these

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14. If 3127^{11859} is multiplied out, then the units' digit in the final product is:

(A) 2 (B) 3 (C) 7 (D) 9 (E) none of these

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15. If $\sqrt{\dfrac{x}{y}\sqrt{\dfrac{y}{x}\sqrt{\dfrac{x}{y}}}} = \left(\dfrac{x}{y}\right)^z$, then z is:

(A) 1/8 (B) 1/4 (C) 3/4 (D) 3/8 (E) none of these

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16. The volume of a right circular cylinder with radius r and height h is V. If the radius is increased by 25% and the height is decreased by 20%, then the volume:

(A) increases by 25% (B) increases by 20% (C) increases by 1/16
(D) remains unchanged (E) none of these

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17. If $f(x) = 3 - x^2$ and $h \neq 0$, then

$$\frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

equals:

(A) -2 (B) $-2x$ (C) 1 (D) $\dfrac{3 - x^2 - 2xh - 3h^2}{h^2}$ (E) $6x - 3h$

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18. The hypotenuse of a right triangle is 1 less than the sum of the legs. If the difference of the legs is 1, then the length of the longest leg is:

(A) $(1 + \sqrt{3})/2$ (B) $(3 + \sqrt{3})/2$ (C) $(3 - \sqrt{3})/2$ (D) $(2 + \sqrt{2})/2$
(E) none of these

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19. Four pennies of diameter d are placed with their centers on the four corners of a $d \times d$ square. The percentage of the area of the square covered by the pennies is:

(A) 60 to 70 (B) 70 to 80 (C) 80 to 90 (D) 90 to 100
(E) none of these

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20. The solution set of the inequality

$$\frac{-3(x-1)(2-x)}{x(x^2+1)} \leq 0$$

is:

(A) $(0, 1] \cup [2, \infty)$ (B) $(0, 2]$ (C) $(-\infty, 1]$ (D) $(-\infty, 0) \cup [2, \infty)$
(E) $(-\infty, 0) \cup [1, 2]$

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21. If $\log_5(\log_4(\log_2 x)) = \log_4(\log_2(\log_5 y)) = \log_2(\log_5(\log_4 z)) = 0$, then $x + y + z$ is equal to:

(A) 890 (B) 945 (C) 1065 (D) 1184 (E) none of these

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22. A circle with radius r units is tangent to sides AB, AD, and CD of rectangle $ABCD$ and passes through the midpoint of diagonal AC. The area of the rectangle (in square units) is:

(A) $3\pi r^2$ (B) $12r$ (C) $2r + r^2$ (D) $8r^2$ (E) $8\pi r$

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23. The value of m that will make the roots of the equation

$$(m^2+1)x^2 - (m+2)x + 1 = 0$$

purely imaginary is:

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

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24. A door has five buttons. To open the door, two buttons must be pressed at the same time, then two more buttons at the same time, and then one button. If no button can be pressed more than once, then the number of possible combinations that could possibly unlock the door is:

(A) 3 (B) 14 (C) 30 (D) 50 (E) 120

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25. If k is a constant, then the remainder upon dividing $x^3 - 2kx^2 + x + 1$ by $x - 5$ is

(A) 0 (B) $131 - 50k$ (C) 81 (D) $-129 + 50k$ (E) k

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26. The value of k that satisfies the equation $\log_{10} 25 = k \log_{100} 25$ is:

(A) $1/2$ (B) 2 (C) 3 (D) $3/4$ (E) none of these

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27. If $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for $|x| \leq 1$, then one can show that π is:

(A) 3.14 (B) $22/7$ (C) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$
(D) $4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$ (E) none of these

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28. If the equation $x^2 + 2dx + b = 0$ has exactly one solution, then this solution is:

(A) -1 (B) $\frac{b}{2a}$ (C) $\frac{-b}{2a}$ (D) $-d$ (E) none of these

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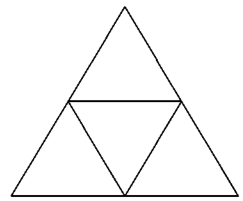
29. An equilateral triangle is formed from four equilateral triangles as shown. If the area of one small triangle is $10\sqrt{3}$ square feet, then the perimeter of the large triangle is:

(A) 36 ft
(B) $9\sqrt{3}$ ft
(C) $12\sqrt{10}$ ft
(D) $6\sqrt{3}$ ft
(E) none of these



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30. An equation of the line that passes through the points of intersection of the graphs of $y = 2x^2 + 3x + 4$ and $y = 14 + 8x - 3x^2$ is:

(A) $y = 5x + 8$ (B) $y = -5x + 2$ (C) $y = 9x + 1$ (D) $y = -9x - 9$
(E) none of these

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31. If $f(x) = 3x + 4$ and $g(x) = 1/x$, then $(g \circ f)^{-1}(x)$ equals:

(A) $\dfrac{1-4x}{3x}$ (B) $\dfrac{x-4}{3}$ (C) $\dfrac{3}{x-4}$ (D) $\dfrac{1}{x}$ (E) none of these

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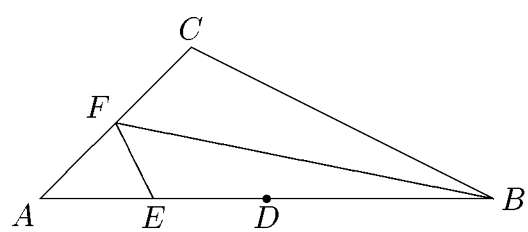
32. In $\triangle ABC$, F is the midpoint of $\overline{AC}$, D is the midpoint of $\overline{AB}$, and E is the midpoint of $\overline{AD}$. If the area of $\triangle BEF$ is 100 square units, then the area of $\triangle ABC$ in square units is:

(A) $266.\overline{6}$
(B) $100\sqrt{2}$
(C) $216\sqrt{3}$
(D) 244
(E) none of these



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33. The solution set of the inequality $\dfrac{3x-5}{2x+1} \leq 1$ is:

(A) $\{x \mid -1/2 < x \leq 6\}$ (B) $\{x \mid 0 \leq x < 5\}$
(C) $\{x \mid -1/2 \leq x \leq 5/3\}$ (D) $\{x \mid 3/2 < x < 6\}$
(E) $\{x \mid x \leq -1/2 \text{ or } x > 6\}$

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34. The values of a that satisfy the equation $\log_4\left(\frac{3}{2}a^2 + \frac{3}{2}a - 2\right) = 2$ are:

(A) $\dfrac{-3/2 \pm \sqrt{12}}{3}$ (B) $-4, 5$ (C) $\dfrac{-3/2 \pm \sqrt{3}}{2}$ (D) ± 5
(E) none of these

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35. If $2x + \sqrt{x} = 1$, then x:

(A) can have either of two different values (B) is an integer (C) is irrational
(D) is rational but not an integer (E) is imaginary

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36. A box contains chips, each of which is red, white, or blue. The number of blue chips is at least one third the number of white chips and at most one fourth the number of red chips. The number that are white or blue is at least 65. The minimum number of red chips possible is:

(A) 45 (B) 54 (C) 68 (D) 76 (E) none of these

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37. Let $x \geq 0$ and suppose $x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}} = 2$, then x equals:

(A) $\sqrt{2}$ (B) 0 (C) 2 (D) 1/2 (E) $\ln 2$

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38. If $4^{2x} + 16 = 17 \cdot 4^x$, then the value of $x^2 + 2x + 5$ is:

(A) 2 (B) 5 (C) 0 or 2 (D) 5 or 13 (E) none of these

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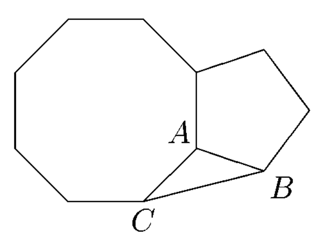
39. A regular octagon and a regular pentagon are positioned as shown. The measure of $\angle ABC$ is:

(A) 31.5°
(B) 30°
(C) 27°
(D) 22.5°
(E) none of these



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40. A parabola with equation $y = ax^2 + bx + c$ has vertex (3,1) and passes through the point (2,0). Then the product abc equals:

(A) 60 (B) -60 (C) 48 (D) -48 (E) none of these

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41. The value of k that satisfies the equation $\log_{100} \sqrt[m]{a^n} = k \log_{1000} a$ is:

(A) m/n (B) n/m (C) $3n/m$ (D) $n/2m$ (E) none of these

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42. If p is prime and if both of the zeros of $x^2 + px - 78p$ are integers, then:

(A) $0 < p < 10$ (B) $10 < p < 20$ (C) $20 < p < 30$ (D) $30 < p < 40$
(E) none of these

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43. Inside a square $ABCD$ with sides of length 12 inches, a line segment PQ is drawn where P is on side AB and 4 inches from A and Q is on side CD and 3 inches from C. (A and C are opposite vertices.) The perpendicular bisector of PQ is drawn and intersects AD, PQ, and BC at X, M, and Y respectively. The ratio of segment MX to MY is:

(A) $5:12$ (B) $5:13$ (C) $12:13$ (D) $13:11$ (E) none of these

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44. Suppose each of three dice has equally likely possible outcomes of $1, 2, 3, 4, 5, 6$. The probability of rolling a sum of ten with the three dice is:

(A) $\frac{5}{72}$ (B) $\frac{7}{72}$ (C) $\frac{25}{216}$ (D) $\frac{1}{8}$ (E) $\frac{1}{6}$

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45. If x, y, and z are assumed to be positive and

$$\begin{aligned} yz &= a(2z + y) \\ xz &= a(2z + x) \\ xy &= a(2x + 2y) \\ 2xz + 2yz + xy &= 12, \end{aligned}$$

then the value of x is:

(A) 1 (B) 2 (C) $\sqrt{5}$ (D) 3 (E) unable to be determined

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46. The solution set for the equation $x^{\log_{10} x} = 100x$ is:

(A) $\{100\}$ (B) $\emptyset$ (C) $\{1/100\}$ (D) $\{.01, 10\}$ (E) none of these

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47. Consider the sets of consecutive even integers $\{2\}$, $\{4, 6\}$, $\{8, 10, 12\}$, $\ldots$, where each set contains one more element than the preceding one and where the first element in each set is two more than the last element of the preceding set. Let S_n be the sum of the elements in the n^{th} set. Then S_{21} equals:

(A) 2226 (B) 9282 (C) 10,164 (D) 12,422 (E) none of these

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48. The number of real solutions of the equation

$$\sqrt{x-4}+\sqrt{x+4}=\sqrt[3]{12x+4}$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) infinite

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49. The area in square units of a rhombus for which one side has length 10 units and the diagonals differ by 4 units is:

(A) 27 (B) 49 (C) 64 (D) 88 (E) 96

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50. The value $\cos(\pi/24)$ is:

(A) $\sqrt{\frac{1+\pi}{24}}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{\sqrt{2+\sqrt{3}}}{2}$ (D) $\frac{\sqrt{3+\sqrt{2+\sqrt{2}}}}{3}$

(E) $\frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}$

Answer Key

1. E	18. B	35. D
2. B	19. B	36. C
3. B	20. E	37. A
4. C	21. C	38. D
5. D	22. D	39. A
6. C	23. A	40. C
7. E	24. C	41. E
8. A	25. B	42. B
9. C	26. B	43. D
10. C	27. D	44. D
11. D	28. D	45. B
12. C	29. C	46. E
13. D	30. A	47. B
14. B	31. A	48. B
15. D	32. A	49. E
16. A	33. A	50. E
17. A	34. E	