Mathematics Competition

Indiana University of Pennsylvania

1999

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.

2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.

3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.

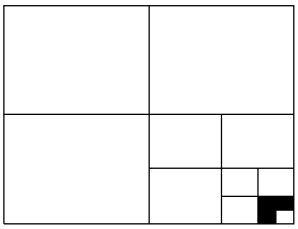
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not expected to finish it.

5. Use of pencil, eraser, and scratch paper only are permitted.

6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. Consider a $12' \times 16'$ rectangle. Divide the rectangle into four equal parts and repeat this with one section three more times as shown. The area of the shaded region is:



(A) 0.75 ft^2
(B) 2.25 ft^2
(C) 3 ft^2
(D) 4.5 ft^2
(E) 9 ft^2

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2. There are five students in a class that achieved a mean score of 78 on its first test. If the first four students had a mean of 73, then the score of the fifth student was:

(A) 73 (B) 98 (C) 92 (D) 78 (E) none of these

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3. Suppose you have four Scrabble tiles with the letters "D", "R", "E", and "E". If you randomly place the tiles in a row, then the probability that you will spell the word DEER is:

(A) 1/12 (B) 1/24 (C) 1/6 (D) 1/4 (E) 1/8

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4. The expression $\sqrt{a^2 - 2ab + b^2}$ is equal to:

(A) $a - b$ (B) $b - a$ (C) $|b - a|$ (D) $(a - b)^2$ (E) none of these

---

5. The expression $(1 - \sin^2 A)(1 + \tan^2 A)$ is equal to:

(A) $\cos^2 A$ (B) $\cot^2 A - 1$ (C) 1 (D) 0 (E) none of these

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6. Of the following statements, the only one that is not true is:

(A) If $x \in \mathbb{R}$, then $|x| = \sqrt{x^2}$.
(B) If x and y are positive real numbers, then $\sqrt{xy} \leq \dfrac{x + y}{2}$.
(C) If $x, y \in \mathbb{R}$, then $|x + y| \leq |x| + |y|$.
(D) If $x \in \mathbb{R}$, then $-|x| \leq x \leq |x|$.
(E) If $x^2 = y^2$, then $x = y$.

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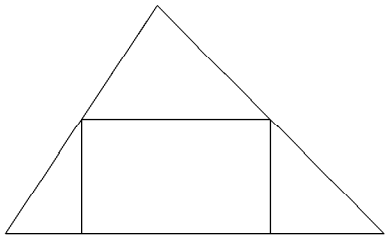
7. Given a triangle, connect the midpoints of two sides of the triangle and drop perpendiculars from the midpoints to the third side of the triangle as shown in the diagram. The ratio of the area of the rectangle to the area of the triangle is:



(A) 2/3
(B) 1/4
(C) 1/2
(D) impossible to determine
(E) none of these

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8. The solution set of the inequality $\dfrac{(x-2)^2}{x-5} \geq 0$ is:

(A) $\{x \mid x = 2 \text{ or } x > 5\}$ (B) $\{x \mid x = 2 \text{ or } x = 5\}$ (C) $\{x \mid 2 \leq x < 5\}$
(D) $\{x \mid x \leq 2 \text{ or } x > 5\}$ (E) $\{\ \}$

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9. The first three terms of an arithmetic progression are $2x - 3$, $2x + 3$, $4x + 5$, in that order. The value of x is:

(A) -3 (B) 0 (C) 3 (D) 6 (E) none of these

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10. A solution for the equation $\log_3(x + 1) - \log_3(x) = 2$ is:

(A) $\dfrac{1}{7}$ (B) $\dfrac{1}{8}$ (C) $\dfrac{\sqrt{37} - 1}{2}$ (D) $\dfrac{\sqrt{33} - 1}{2}$ (E) none of these

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11. The polynomial $x^2 + 3x + 4$ is a factor of $x^4 + 8x^3 + px^2 + qx + 24$ if p and q have values, respectively, of:

(A) 20 and 30 (B) 22 and 34 (C) 14 and 15 (D) 38 and 25
(E) none of these

---

12. The diameter of a circle with radius 3 in is the side of a square. The exact area of the part of the square that lies outside the circle is:

(A) $36 - 9\pi$ in^2 (B) 21.87 in^2 (C) 7.74 in^2 (D) $36 - 9\pi/2$ in^2
(E) none of these

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13. If r_1 and r_2 are the roots of the equation $3x^{-2} + 4x^{-1} + 1 = 0$, then $r_1 + r_2 - r_1 r_2$ is equal to:

(A) -1 (B) -3 (C) -4 (D) -7 (E) 0

---

14. If $f(0) = 4$, $g(x) = 2f(x-2) + 4$, and $h(x) = \sqrt{x^2 + 25}$, then $(h \circ g)(2)$ is equal to:

(A) $\sqrt{29}$ (B) 4 (C) 6 (D) 144 (E) 13

---

15. The number of ordered pair solutions of the system

$$\begin{aligned} 2x^3 - 2y &= 4 \\ 9x - 3y &= 12 \end{aligned}$$

is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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16. The set of all x satisfying $\sin^2 x + \sin x = 2$ with $0 \leq x < 2\pi$ is:

(A) $\{\pi/2\}$ (B) $\{\pi/4\}$ (C) $\{\pi/2, 3\pi/2\}$ (D) 1 (E) none of these

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17. Three pairs of blue socks, two pairs of black socks, and five pairs of brown socks are thrown into a drawer unsorted. If you were to choose two socks without turning on the lights, the probability that you would get a matching pair is:

(A) 1/19 (B) 33/95 (C) 17/19 (D) 1/2 (E) none of these

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18. The radiator in a car is filled with a solution of 70% antifreeze and 30% water. The manufacturer's recommended solution for optimal cooling is 50% antifreeze and 50% water. If the car's radiator capacity is 4.2 liters, the number of liters of the present coolant solution that should be drained and replaced with water to reduce the antifreeze concentration to the recommended level is:

(A) 4.2 (B) 3.2 (C) 2.2 (D) 1.2 (E) .2

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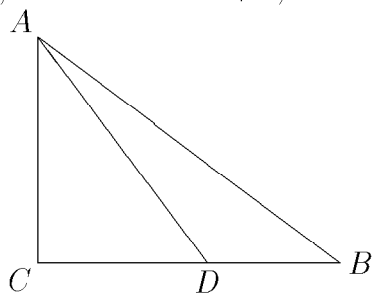
19. In the right triangle shown, if $AB = 20$, $AC = 12$, and $AD = DB + 8$, then CD is:

(A) 14
(B) $12\sqrt{2}$
(C) 8
(D) 9
(E) none of these



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20. The number of terms in the simplified expansion of $[(a+3b)^4(a-3b)^4]^2$ is:

(A) 6 (B) 7 (C) 8 (D) 10 (E) none of these

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21. The solution set of $(\log_{10} x)^3 = \log_{10} x^9$ is:

(A) $\{1000\}$ (B) $\{.001\}$ (C) $\{1\}$ (D) $\{2\}$ (E) none of these

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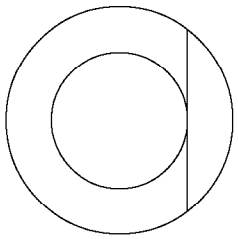
22. In the accompanying figure, the longest straight line that can be drawn on a circular track is 300 feet. The area of the track in square feet is:



(A) $300^2\pi$
(B) $150^2\pi$
(C) 300π
(D) impossible to determine
(E) none of these

---

23. The sum of three numbers is 70. If the ratio of the first to the second is 3/4 and the ratio of the second to the third is 4/7, then the second number is:

(A) 12 (B) 16 (C) 20 (D) 24 (E) none of these

---

24. The final digit in the number 7^{459} is:

(A) 0 (B) 7 (C) 9 (D) 3 (E) 1

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25. The solution to the inequality $2x + 1 \leq 4x - 3 \leq x + 7$ is:

(A) $x \leq 2$ or $x \geq 10/3$ (B) $2 \leq x \leq 10/3$ (C) $2 < x < 3$ (D) $x > 10/3$
(E) $x \leq 2$

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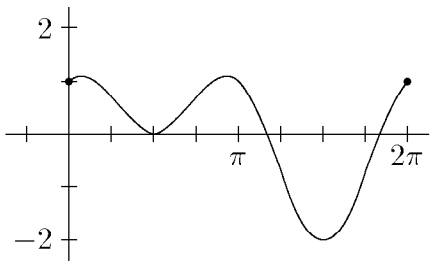
26. Consider the following graph of a function defined on $[0, 1]$. The function is given by:



(A) $\sin x + \cos 2x$
(B) $\sin x - \cos 2x$
(C) $\cos 2x - \sin x$
(D) $\sin 2x + \cos x$
(E) $\cos x - \sin 2x$

---

27. Each side of triangle ABC has length 24 units. Let D be the foot of the perpendicular dropped from A onto BC, and let E be the midpoint of AD. The length of BE in units is:

(A) $\sqrt{210}$ (B) 15 (C) $\sqrt{241}$ (D) $\sqrt{252}$ (E) 16

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28. If the roots of $x^2 + bx + c = 0$ are r and s, then $r^2 + s^2$ is equal to:

(A) $b^2 - 2c$ (B) $2b^2 - 4c$ (C) $3b^2 - 2c$ (D) $b^2 + 2c$ (E) none of these

---

29. The value of x so that $\log_{10}(10 \log_{10}(\log_{10} x^{-10})) = 1$ is:

(A) 100 (B) .001 (C) .5 (D) .1 (E) none of these

---

30. James and Wendy can paint a house in three-fourths the time that it takes James working alone. Wendy takes 12 days to paint a house alone. The rate in houses per day at which James paints alone is:

(A) 2 (B) 1/2 (C) 4 (D) 1/4 (E) 1/6

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31. If $f(x) = \sqrt{\dfrac{x^3 - 3x^2 + 4x - 12}{x(x^2 + 1)}}$, then the domain of f is:

(A) $(-\infty, 0) \cup (3, \infty)$ (B) $(0, 3]$ (C) $(-\infty, 0) \cup [3, \infty)$
(D) $(-\infty, 0) \cup (0, 3]$ (E) $(-\infty, \infty)$

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32. Suppose two circles, each of radius 1/2, intersect so that each circle goes through the center point of the other circle. The length of the line segment between the two circle intersection points is:

(A) $\dfrac{\sqrt{3}}{2}$ (B) $\dfrac{3}{4}$ (C) $\dfrac{\sqrt{2}}{2}$ (D) $\dfrac{2}{3}$ (E) $\dfrac{1}{2}$

---

33. The solution in interval notation to the inequality $|x| > |x - 2|$ is:

(A) $(0, 2)$ (B) $(-\infty, 0) \cup (2, \infty)$ (C) $(1, \infty)$ (D) $(1, 2)$
(E) $(-\infty, 0) \cup (1, \infty)$

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34. The solution set of $x^2 - \cos x + 1 = 0$ is:

(A) $\{-1\}$ (B) $\{.5\}$ (C) $\{\sqrt{2}\}$ (D) $\{2\sqrt{2}\}$ (E) none of these

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35. For one root of $ax^2 + bx + c = 0$ to be four times the other root, the coefficients a, b, and c must be related as follows:

(A) $4b^2 = 9c$ (B) $2b^2 = 9ac$ (C) $4b^2 = 25ac$ (D) $b^2 = 8ac$
(E) $9b^2 = 2ac$

---

36. A number that when divided by 126 leaves a remainder of 125, when divided by 75 leaves a remainder of 74, when divided by 24 leaves a remainder of 23, and when divided by 20 leaves a remainder of 19 is:

(A) 10,421 (B) 11,624 (C) 12,601 (D) 13,714 (E) none of these

---

37. The distance from the point $P = (3, 4)$ to the line ℓ given by the equation $x - 5y = 4$ is between:

(A) 1 and 2 (B) 2 and 3 (C) 3 and 4 (D) 4 and 5 (E) 5 and 6

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38. The number of real solutions of the equation $|3x - |2x - 1|| = 1$ is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

---

39. If $p \log_{10} m = b - \log_{10} n$, then m is equal to:

(A) $\dfrac{10^b}{n}$ (B) $\sqrt{10b}$ (C) $\dfrac{10^b}{pn}$ (D) $\sqrt{\dfrac{10^b}{n}}$ (E) none of these

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40. The solution in interval notation to the inequality

$$\frac{1}{x} + \frac{1}{x-1} \geq \frac{2}{x-2}$$

is:

(A) $(-\infty, 0) \cup (2/3, 1) \cup (2, \infty)$ (B) $[2/3, 1) \cup (2, \infty)$ (C) $(0, 2/3) \cup (1, 2)$
(D) $(-\infty, \infty)$ (E) $(0, 2/3] \cup (1, 2)$

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41. The inverse of the function $f(x) = \dfrac{x+4}{2x-3}$ is:

(A) $f^{-1}(x) = \dfrac{x+1}{x-3}$ (B) $f^{-1}(x) = \dfrac{2x-3}{3x+5}$ (C) $f^{-1}(x) = \dfrac{3x+4}{2x-1}$

(D) $f^{-1}(x) = \dfrac{x}{x+2}$ (E) $f^{-1}(x) = \dfrac{x+5}{x}$

---

42. A circle is inscribed inside an equilateral triangle which in turn is inscribed inside another circle. The ratio of the area of the smaller circle to that of the larger circle is:

(A) $3:5$ (B) $1:4$
(C) $1:2$ (D) impossible to determine from the information given
(E) none of these

---

43. Of the following quantities, the only one that is a solution to $x^{15} - 3x^{10} + 3x^5 - 3 = 0$ is:

(A) $\sqrt[5]{1+\sqrt[3]{2}}$ (B) $\sqrt[5]{1+\sqrt[3]{3}}$ (C) $\sqrt[5]{1+\sqrt[3]{4}}$ (D) $\sqrt[5]{1+\sqrt[3]{5}}$
(E) $\sqrt[5]{1+\sqrt[3]{6}}$

---

44. If the number $15! = 15 \cdot 14 \cdot 13 \cdot \cdots \cdot 3 \cdot 2 \cdot 1$ ends in k zeros when written in base 12 and h zeros when written in base 20, then $k + h$ equals:

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

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45. The number of real solutions of the equation $x^2 - xe^x - x\ln x + e^x \ln x = 0$ is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

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46. The solution to the inequality $|x-2| < |x-1| < |x|$ is:

(A) $x > 1/2$ (B) $1/2 < x < 3/2$ (C) $x > 2$ (D) $x > 3/2$ (E) $x > 1$

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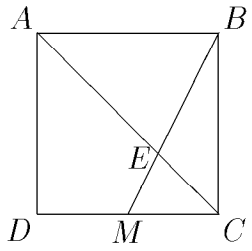
47. Given the square $ABCD$ with M the midpoint of $\overline{DC}$, the ratio of the area of the triangle MEC to that of the quadrilateral $AEMD$ is:

(A) $1:3$
(B) $1:4$
(C) $1:5$
(D) $2:3$
(E) none of these



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48. If $\log_a b = 7$ and $\log_b d = -5$, then

$$\log_a\left(\frac{\sqrt[4]{b^3}}{d^2}\right) - \log_b\left(\frac{\sqrt[4]{d^3}}{b^2}\right)$$

is equal to:

(A) $\dfrac{7b}{4d}$ (B) a (C) $\log_a 4 - \log_a 3$ (D) -12 (E) 81

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49. The value of $\left(52 + 6\sqrt{43}\right)^{3/2} - \left(52 - 6\sqrt{43}\right)^{3/2}$ is:

(A) $158 + \sqrt{43}$ (B) 828 (C) $27\sqrt{43} + 426$ (D) $104\sqrt{43} + 129$
(E) 652

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50. The range of the function $f(x) = \dfrac{\sqrt{x-1}}{x+2}$ is:

(A) $[0, 1)$ (B) $\left[0, \sqrt{3}/6\right]$ (C) $\left[0, \sqrt{2}/5\right]$ (D) $[0, \infty)$ (E) $(-\infty, \infty)$

Answer Key

1. B
2. B
3. A
4. C
5. C
6. E
7. C
8. A
9. E
10. B
11. E
12. D
13. D
14. E
15. C
16. A
17. B
18. D
19. D
20. E
21. E
22. B
23. C
24. D
25. B
26. A
27. D
28. A
29. D
30. D
31. C
32. A
33. C
34. E
35. C
36. E
37. D
38. C
39. E
40. E
41. C
42. B
43. A
44. D
45. A
46. D
47. C
48. E
49. B
50. B