MATH 445/545 Homework 3: Due April 7th, 2016

Answer the following questions. Please include the question with the solution (write or type them out doing this will help you digest the problem). Please give me: Question, Answer, Conclusion, do not just included a copy of this document. Note the problems are separated into two sections a set for all students and a bonus set for those taking the course at the 545 level.

All Students

1. Use the simplex method in matrix form to solve the following primal linear program:

   maximize \( z = 2x_1 - x_2 + x_3 \)
   subject to \( 3x_1 + x_2 + x_3 \leq 60 \)
   \( x_1 - x_2 + 2x_3 \leq 10 \)
   \( x_1 + x_2 - x_3 \leq 20 \)
   \( x_1, x_2, x_3 \geq 0. \)

   Take the dual of the above linear program and again use the matrix simplex method to solve the problem.

   Note: for a problem in the standard form given in class recall that any current tableau in the simplex algorithm can be represented in matrix form using the basis as follows:

   \[
   \begin{array}{cccccccc}
   x_N^T & x_B^T & \text{RHS} & \text{Basis} \\
   -c_N^T + c_B^T B^{-1}N & 0 & c_B^T B^{-1}b & z \\
   B^{-1}N & I & B^{-1}b & x_B
   \end{array}
   \]

   Solution:

   Lets first solve the primal problem using a tableau so we have something to use as comparison.

   \[
   \begin{array}{ccccccccccc}
   \text{Row} & z & x_1 & x_2 & x_3 & s_1 & s_1 & s_3 & \text{RHS} & \text{Basis} \\
   0 & 1 & -2 & 1 & -1 & 0 & 0 & 0 & 0 & z \\
   1 & 0 & 3 & 1 & 1 & 1 & 0 & 0 & 60 & s_1 \\
   2 & 0 & 1 & -1 & 2 & 0 & 1 & 0 & 10 & s_2 \\
   3 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 20 & s_3 \\
   \end{array}
   \]

   Choose \( x_1 \) to enter in place of \( s_2 \).

   \[
   \begin{array}{ccccccccccc}
   \text{Row} & z & x_1 & x_2 & x_3 & s_1 & s_1 & s_3 & \text{RHS} & \text{Basis} \\
   0 & 1 & 0 & -1 & 3 & 0 & 2 & 0 & 20 & z \\
   1 & 0 & 0 & 4 & -5 & 1 & -3 & 0 & 30 & s_1 \\
   2 & 0 & 1 & -1 & 2 & 0 & 1 & 0 & 10 & x_1 \\
   3 & 0 & 0 & 2 & -3 & 0 & -1 & 1 & 10 & s_3 \\
   \end{array}
   \]
Choose $x_2$ to enter in place of $s_3$.

<table>
<thead>
<tr>
<th>Row</th>
<th>$z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>RHS</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1.5</td>
<td>0</td>
<td>1.5</td>
<td>0.5</td>
<td>25</td>
<td>$z$</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
<td>10</td>
<td>$s_1$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.5</td>
<td>0</td>
<td>0.5</td>
<td>0.5</td>
<td>15</td>
<td>$x_1$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1.5</td>
<td>0</td>
<td>-0.5</td>
<td>0.5</td>
<td>5</td>
<td>$s_3$</td>
</tr>
</tbody>
</table>

Here we have achieved the optimal solution with $x_1 = 15$, $x_2 = 5$, and $x_3 = 0$ the optimal objective is 25.

For this problem we make the following definitions in order to solve the problem using the matrix simplex method:

$$
\mathbf{x} = \begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    s_1 \\
    s_2 \\
    s_3
\end{pmatrix}, \quad
\mathbf{c} = \begin{pmatrix}
    2 \\
    -1 \\
    1 \\
    0 \\
    0 \\
    0
\end{pmatrix}, \quad
\mathbf{b} = \begin{pmatrix}
    60 \\
    10 \\
    20
\end{pmatrix}
$$

and

$$
\mathbf{A} = \begin{pmatrix}
    3 & 1 & 1 & 1 & 0 & 0 \\
    1 & -1 & 2 & 1 & 0 & 1 \\
    1 & 1 & -1 & 0 & 0 & 1
\end{pmatrix}.
$$

In order to implement the simplex algorithm in matrix format we break the given coefficient matrix $\mathbf{A}$ into to parts: a basic part and a non-basic part. Thus,

$$
\mathbf{A} = [\mathbf{N}|\mathbf{B}]
$$

In matrix form we start by defining $\mathbf{x_B} = (s_1 \ s_2 \ s_3)^T$, and $\mathbf{x_N} = (x_1 \ x_2 \ x_3)^T$ we have:

$$
\mathbf{N} = \begin{pmatrix}
    3 & 1 & 1 \\
    1 & -1 & 2 \\
    1 & 1 & -1
\end{pmatrix}, \quad
\mathbf{B} = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{pmatrix} \implies \mathbf{B}^{-1} = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{pmatrix}.
$$

with

$$
\mathbf{c_N}^T = (2 \ -1 \ 1) \\
\mathbf{c_B}^T = (0 \ 0 \ 0)
$$
Evaluating each of the expressions in the tableau we have:

\[-c_N^T + c_B^T B^{-1} N = \begin{pmatrix} -2 & 1 & -1 \end{pmatrix}
\]
\[c_B^T B^{-1} b = \begin{pmatrix} 0 \end{pmatrix}
\]
\[B^{-1} b = \begin{pmatrix} 60 \\ 10 \\ 20 \end{pmatrix}
\]
\[B^{-1} N = \begin{pmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{pmatrix}
\]

Note that the non-basic variable \(x_1\) should be picked to enter the basis. Doing the ratio test using the first column of \(B^{-1} N\) and the current right hand side vector \(B^{-1} b\):

\[\min \left\{ \frac{60}{3} = 20, \frac{10}{1} = 10, \frac{20}{1} = 20 \right\}
\]

and we choose \(s_2\) to leave the basis. This leads to the new values for \(x_B = (s_1 \ x_1 \ s_3)^T\), and \(x_N = (s_2 \ x_2 \ x_3)^T\) we have:

\[N = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \Rightarrow \quad B^{-1} = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.
\]

with

\[c_N^T = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix}
\]
\[c_B^T = \begin{pmatrix} 0 & 2 & 0 \end{pmatrix}
\]

Evaluating each of the expressions in the tableau we have:

\[-c_N^T + c_B^T B^{-1} N = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix}
\]
\[c_B^T B^{-1} b = \begin{pmatrix} 20 \end{pmatrix}
\]
\[B^{-1} b = \begin{pmatrix} 30 \\ 10 \\ 10 \end{pmatrix}
\]
\[B^{-1} N = \begin{pmatrix} -3 & 4 & -5 \\ 1 & -1 & 2 \\ -1 & 2 & -3 \end{pmatrix}
\]
Note that the non-basic variable $x_2$ should be picked to enter the basis. Doing the ratio test using the second column of $B^{-1}N$ and the current right hand side vector $B^{-1}b$:

$$\min \left\{ \frac{30}{4} = 7.5, \frac{10}{2} = 5 \right\}$$

and we choose $s_3$ to leave the basis. This leads to the new values for $x_B = (s_1 \ x_1 \ x_2)^T$, and $x_N = (s_2 \ s_3 \ x_3)^T$ we have:

$$N = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \implies B^{-1} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$ 

with

$$c_N^T = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad c_B^T = \begin{pmatrix} 0 & 2 & -1 \end{pmatrix}$$

Evaluating each of the expressions in the tableau we have:

$$-c_N^T + c_B^T B^{-1}N = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$c_B^T B^{-1}b = \begin{pmatrix} 25 \end{pmatrix}$$

$$B^{-1}b = \begin{pmatrix} 10 \\ 15 \\ 5 \end{pmatrix}$$

$$B^{-1}N = \begin{pmatrix} -1 & -2 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

We note here that none of the reduced costs given in $-c_N^T + c_B^T B^{-1}N$ are negative, and we have reached the optimal solution $z = 25$ when

$$s_1 = 10, x_1 = 15, \text{ and } x_2 = 5.$$
and some minor adjustment gives:

\[
\begin{align*}
\text{minimize} \quad & w = 60y_1 + 10y_2 + 20y_3 \\
\text{subject to} \quad & 3y_1 + y_2 + y_3 \geq 2 \\
& -y_1 + y_2 - y_3 \leq 1 \\
& y_1 + 2y_2 - y_3 \geq 1 \\
& y_1, y_2, y_3 \geq 0
\end{align*}
\]

Adding slack and excess variables we can place this LP into matrix form as follows:

\[
x = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ e_1 \\ s_2 \\ e_3 \end{pmatrix}, \quad c = \begin{pmatrix} 60 \\ 10 \\ 20 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}
\]

and

\[
A = \begin{pmatrix} 3 & 1 & 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & -1 \end{pmatrix}
\]

Adjust the right hand side and the matrix \(A\) in order to reflect that we need a starting basic feasible solution and we can use the dual simplex method in order to deal with any "infeasibility" that may arise on the RHS. Thus,

\[
x = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ e_1 \\ s_2 \\ e_3 \end{pmatrix}, \quad c = \begin{pmatrix} 60 \\ 10 \\ 20 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}
\]

and

\[
A = \begin{pmatrix} -3 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ -1 & -2 & 1 & 0 & 0 & 1 \end{pmatrix}
\]

In order to implement the simplex algorithm in matrix format we break the given coefficient matrix \(A\) into parts: a basic part and a non-basic part. Thus,

\[
A = [N|B]
\]

In matrix form we start by defining \(x_B = (e_1 \ s_2 \ e_3)^T\), and \(x_N = (y_1 \ y_2 \ y_3)^T\) we have:
\[ N = \begin{pmatrix} -3 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

with

\[ c_N^T = \begin{pmatrix} 60 & 10 & 20 \end{pmatrix}, \quad c_B^T = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}. \]

Evaluating each of the expressions in the tableau we have:

\[ -c_N^T + c_B^T B^{-1} N = \begin{pmatrix} -60 & -10 & -20 \end{pmatrix}, \quad c_B^T B^{-1} b = \begin{pmatrix} 0 \end{pmatrix}, \quad B^{-1} b = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \quad B^{-1} N = \begin{pmatrix} -3 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -2 & 1 \end{pmatrix}. \]

Conduct a pivot of dual simplex we see that \( e_1 \) should leave the basis, and \( y_2 \) should enter in its place.

\[ N = \begin{pmatrix} -3 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \implies B^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}. \]

with

\[ c_N^T = \begin{pmatrix} 60 & 0 & 20 \end{pmatrix}, \quad c_B^T = \begin{pmatrix} 10 & 0 & 0 \end{pmatrix}. \]

Evaluating each of the expressions in the tableau we have:

\[ -c_N^T + c_B^T B^{-1} N = \begin{pmatrix} -30 & -10 & -10 \end{pmatrix}, \quad c_B^T B^{-1} b = \begin{pmatrix} 20 \end{pmatrix}, \quad B^{-1} b = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad B^{-1} N = \begin{pmatrix} 3 & -1 & 1 \\ -4 & 1 & -2 \\ 5 & -2 & 3 \end{pmatrix}. \]
Here we see that we still need one more pivot of dual simplex. Where we aim to get $s_2$ out of the basis in place $y_3$ in its place (via the ratio test).

\[
N = \begin{pmatrix} -3 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ -2 & 1 & 1 \end{pmatrix} \quad \implies \quad B^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}.
\]

with

\[
c^N_T = \begin{pmatrix} 60 & 0 & 0 \end{pmatrix}, \quad c^B_T = \begin{pmatrix} 10 & 20 & 0 \end{pmatrix}
\]

Evaluating each of the expressions in the tableau we have:

\[
-c^N_T + c^B_T B^{-1}N = \begin{pmatrix} -10 & -15 & -5 \end{pmatrix}, \quad c^B_T B^{-1}b = \begin{pmatrix} 25 \end{pmatrix}
\]

\[
B^{-1}b = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad B^{-1}N = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 2 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}
\]

This is a min problem so we have found an optimal solution with

\[
w = 25, \text{ and } y_1 = 0, y_2 = \frac{3}{2}, y_3 = \frac{1}{2}
\]

confirming the shadow prices in the optimal primal tableau, and the optimal objective function value form the primal problem solution.

2. The following tableau for a maximization problem was obtained when solving a linear program:

<table>
<thead>
<tr>
<th>Row</th>
<th>$z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>RHS</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$a$</td>
<td>0</td>
<td>$b$</td>
<td>$c$</td>
<td>3</td>
<td>$d$</td>
<td>$z$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>$e$</td>
<td>0</td>
<td>2</td>
<td>$f$</td>
<td>$x_3$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>$g$</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$x_1$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$h$</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>$x_5$</td>
</tr>
</tbody>
</table>

Find conditions on the parameters $a, b, \ldots, h$ so that the following statements are true. (In the solution parameters that can take on any value will not be mentioned).
(a) The current basis is optimal.

Solution:

Here the current basis is optimal if:

- $c := 0$ as $x_5$ is basic.
- $a$ and $b$ need to be non-negative making no non-basic variable attractive.

(b) The current basis is the unique optimal basis.

Solution:

- $c := 0$ as $x_5$ is basic.
- $a$ and $b$ must be positive to guarantee no alternate optimal solutions.

(c) The current basis is optimal but alternative optimal bases exist.

Solution:

- $c := 0$ as $x_5$ is basic.
- Either $a$ and $b$ must be zero with the following sub-cases.
  - If $a = 0$ then for $g > 0$ would give an alternate optimal solution. When $g \leq 0$ the ratio test would not be restrictive and there would be a direction of alternate optimal solutions.
  - If $b = 0$ then for when either $e$ or $h$ or both are positive we would have an alternate optimal solution. If both $e$ and $h$ are negative then ratio test would not be restrictive and there would be a direction of alternate optimal solutions.

(d) The problem is unbounded.
Solution:
Assume again that \( c = 0 \) as \( x_5 \) is basic. The problem will be unbounded in two cases:

- \( a < 0 \) and \( g \leq 0 \).
- \( b < 0 \) and \( e, h \leq 0 \).

In both cases we would have an attractive variable that would enter into the basis, and the ratio test would not have to restrict the value of the decision variables to be non-negative.

(e) The current solution will improve if \( x_4 \) is increased. When \( x_4 \) is entered into the basis, the change in the objective function is zero.

Solution:
The current solution will improve if \( x_4 \) is increased means that \( b \) will be the most attractive. We will rig the other parameters such that after entering the objective function does not change.

- \( c = 0 \) as \( x_5 \) is basic.
- \( b < 0 \) and \( a \geq 0 \) making \( x_4 \) attractive.
- \( e > 0, f = 0 \) and \( h > 0 \). Will assure \( e \) wins the Ratio Test, and will have no effect on the value of \( d \) when Row 1 is added to Row 0.

545 Additional Homework

1. Use the duality theorems to prove that the system:

\[
\begin{align*}
A^T y & \leq 0 \\
b^T y & > 0
\end{align*}
\]

has a solution if and only if the system:

\[
\begin{align*}
Ax &= b \\
x &\geq 0
\end{align*}
\]

has no solution.
Solution:

First recall the primal dual pair:

**Primal:** \( \min 0 \text{ s.t. } Ax = b, x \geq 0. \)

**Dual:** \( \max b^T y \text{ s.t. } A^T y \leq 0, y - \text{free} \)

and note that weak duality states that for a feasible point in the dual and a feasible point in the primal the following relationship is satisfied:

\[ b^T \leq 0 \]

Assume that \( A^T y \leq 0 \) and \( b^T y > 0 \) has a solution \( y \).

\[ \implies b^T y > 0 \implies (Ax)^T y > 0 \implies x^T A^T y > 0 \]

and as \( A^T y < 0 \) then \( x < 0 \) for the above to hold and the second system can not have a solution.

If the system \( Ax = b, x \geq 0 \) has a solution. Then note that its will have a bound on its objective necessitating that

\[ b^T y \leq 0 \]

and contradicting the possibility \( A^T y \leq 0, b^T y > 0 \) has a solution.