NAME: ____________________________
1. (15 pts) Solve the following linear program graphically, prior to stating the solution write the feasible region as a convex set.

maximize \( z = x_1 + x_2 \)
subject to \( x_1 + x_2 \leq 4 \)
\( 2x_1 + x_2 \geq 5 \)
\( x_1, x_2 \geq 0 \).

**Solution:** We plot the feasible region to the given expressions in the Figure below.

Make note that the feasible region is given by the convex combination of the points:

\( A = (4, 0), B = (1, 3), \) and \( C = (2.5, 0) \)

Thus, letting \( \sigma_1, \sigma_2, \) and \( \sigma_3 \in [0, 1] \) such that \( \sum_{i=1}^{3} \sigma_i = 1 \) then any point in the feasible region, \( D = (x_f, y_f) \) may be written as

\[
(x_f, y_f) = ((\sigma_1(4) + \sigma_2(1) + \sigma_3(2.5)), (\sigma_1(0) + \sigma_2(3) + \sigma_3(0)))
\]

\[
= ((\sigma_1(4) + \sigma_2(1) + \sigma_3(2.5)), \sigma_2(3))
\]

By increasing the countours of the objective function the maximal value of \( z \) can be seen as the convex combination of the points \( A \) and \( B \). Thus, the optimal solution of \( z = 4 \) occurs for any point 0 in the convex set:

\[
O = \{(x, y) | (\alpha 4 + (1 - \alpha)1, \alpha(0) + (1 - \alpha)3)\}
\]

\[
= \{(x, y) | (3\alpha + 1, 3 - 3\alpha)\}
\]

where \( \alpha \in [0, 1] \).
2. (20 pts) Solve the following linear program using the simplex method.

\[
\begin{align*}
\text{maximize} & \quad z = 2x_1 + 3x_2 \\
\text{subject to} & \quad 2x_1 + x_2 \geq 4 \\
& \quad x_1 - x_2 \geq -1 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

**Solution:** Write the program in a more standardized form:

\[
\begin{align*}
\text{maximize} & \quad z = 2x_1 + 3x_2 - Ma_1 \\
\text{subject to} & \quad 2x_1 + x_2 - e1 + a_1 = 4 \\
& \quad -x_1 + x_2 + s_1 = 1 \\
& \quad x_1, x_2, e_1, s_1, a_1 \geq 0.
\end{align*}
\]

We can now write out a starting simplex tableau:

<table>
<thead>
<tr>
<th>Row</th>
<th>z</th>
<th>x1</th>
<th>x2</th>
<th>e1</th>
<th>s1</th>
<th>a1</th>
<th>RHS</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>M</td>
<td>z</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>a1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>s1</td>
<td></td>
</tr>
</tbody>
</table>

We adjust the first tableau so that the basic variable \(a_1\) has an identity column.

<table>
<thead>
<tr>
<th>Row</th>
<th>z</th>
<th>x1</th>
<th>x2</th>
<th>e1</th>
<th>s1</th>
<th>a1</th>
<th>RHS</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>M</td>
<td>z</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>a1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>s1</td>
<td></td>
</tr>
</tbody>
</table>

Pivot \(x_1\) into the basis in place of \(a_1\).

<table>
<thead>
<tr>
<th>Row</th>
<th>z</th>
<th>x1</th>
<th>x2</th>
<th>e1</th>
<th>s1</th>
<th>a1</th>
<th>RHS</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>&quot;M+1&quot;</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1/4</td>
<td>-1/2</td>
<td>1</td>
<td>1/2</td>
<td>2</td>
<td>x1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>3/2</td>
<td>-3</td>
<td>1</td>
<td>1/2</td>
<td>3</td>
<td>s1</td>
</tr>
</tbody>
</table>

Pivot \(x_2\) into the basis. Ratio test

\[
\text{Take minimum of } \frac{2}{3} = 4, \frac{2}{3} = 2
\]

and \(s_1\) leaves the basis giving:

<table>
<thead>
<tr>
<th>Row</th>
<th>z</th>
<th>x1</th>
<th>x2</th>
<th>e1</th>
<th>s1</th>
<th>a1</th>
<th>RHS</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-5/3</td>
<td>4</td>
<td>5/3</td>
<td>M+5/3</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1/3</td>
<td>-3</td>
<td>1/3</td>
<td>1/3</td>
<td>1</td>
<td>x1</td>
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<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1/3</td>
<td>3</td>
<td>1/3</td>
<td>2</td>
<td>x2</td>
</tr>
</tbody>
</table>
The above given tableau has a direction of unboundedness (Look at the column for $e_1$), and there is no maximum value for our linear program. From the tableau we see that the point

$$X = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is feasible. We also have the constraints:

$$x_1 - \frac{1}{3}e_1 = 1$$
$$x_2 - \frac{1}{3}e_1 = 2$$

Here if $x_1$ and $x_2$ are increased by 1 and $e_1$ is increased by 3 the constraints will remain satisfied, and the direction of unboundedness is

$$d = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}.$$

Solutions of the form

$$X_{un} = X + \alpha d$$

for $\alpha > 0$ will be feasible, and increasing the value of $\alpha$ increases the solution without bound.
3. (15 pts) Give a brief description for each of the following:

(a) Explain in words the ratio test in the simplex algorithm and what the ratio test is doing.

**Solution:** Ratio test: Once a non-basic variable is chosen to enter the basis, the ratio of the right hand side value for each constraint over the positive coefficient in the entering variables column is examined. The winner of the test is the constraint corresponding to the minimum outcome in this test. The ratio allows us to see how large we can make the entering variable. Specifically picking the minimum in the test keeps the all of the current basic variable from becoming negative during a pivot of the simplex algorithm.

(b) How can you determine if a linear program using the simplex method is infeasible? Assume you have written the problem in standard form including slack, excess, and artificial variables.

**Solution:** Linear programs will be infeasible if we can not find an initial basic feasible solution for the simplex algorithm. This occurs if we have an artificial variable variable as a “Basic” variable once the simplex algorithm has reached an optimal tableau.

(c) Assuming a maximization problem: State Bland’s rule and explain why it is used.

**Solution:** Bland’s rule is used to keep the simplex algorithm from cycling. Assume that slack and excess variables are numbered $x_{n+1}, x_{n+2}, \ldots$.

- For a maximization problem choose the entering variable with a negative row 0 coefficient that has the smallest index.
- For ties in the ratio test, break the tie by choosing the winner of the ratio test to be the variable with the smallest subscript or index.
4. (10 pts) Change the following to a maximization problem, and put in all the slack, excess, and artificial variables. Write the first tableau for the problem. **Don’t solve.**

\[
\begin{align*}
\text{minimize} \quad & z = 2x_1 + 3x_2 + 2x_3 + 3x_4 \\
\text{subject to} \quad & x_1 + 3x_2 + 5x_3 + 2x_4 \leq 5 \\
& 2x_1 + 3x_3 \geq 6 \\
& 5x_1 + 3x_2 + x_4 = 7 \\
& x_1 \geq 0, x_2 \geq 0, x_4 \geq -2
\end{align*}
\]

**Solution:** Adding slack, excess and artificial variables gives:

\[
\begin{align*}
\text{minimize} \quad & z = 2x_1 + 3x_2 + 2x_3 + 3x_4 + Ma_1 + Ma_2 \\
\text{subject to} \quad & x_1 + 3x_2 + 5x_3 + 2x_4 + s_1 = 5 \\
& 2x_1 + 3x_3 - e_1 + a_1 = 6 \\
& 5x_1 + 3x_2 + x_4 + a_2 = 7 \\
& x_1 \geq 0, x_2 \geq 0, x_4 \geq -2, s_1 \geq 0, a_1 \geq 0, a_2 \geq 0
\end{align*}
\]

Doing two change of variables:

\[x_3 = x_3' - x_3'' \text{ and } x_4' = x_4 + 2 \rightarrow x_4 = x_4' - 2\]

\[
\begin{align*}
\text{minimize} \quad & z + 6 = 2x_1 + 3x_2 + 2x_3' - 2x_3'' + 3x_4' + Ma_1 + Ma_2 \\
\text{subject to} \quad & x_1 + 3x_2 + 5x_3' - 5x_3'' + 2x_4' + s_1 = 9 \\
& 2x_1 + 3x_3' - 3x_3'' - e_1 + a_1 = 6 \\
& 5x_1 + 3x_2 + x_4' + a_2 = 9 \\
& x_1, x_2, x_3', x_3'', x_4', s_1, a_1, a_2 \geq 0
\end{align*}
\]

Change the minimization constraint to a maximization by considering the objective function

\[
\begin{align*}
\text{maximize} \quad & -z - 6 = -2x_1 - 3x_2 - 2x_3' + 2x_3'' - 3x_4' - Ma_1 - Ma_2
\end{align*}
\]

Thus the first tableau for the problem is given by:

<table>
<thead>
<tr>
<th>Row</th>
<th>z</th>
<th>x_1</th>
<th>x_2</th>
<th>x_3'</th>
<th>x_3''</th>
<th>x_4'</th>
<th>s_1</th>
<th>e_1</th>
<th>a_1</th>
<th>a_2</th>
<th>RHS</th>
<th>Basic</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>2</td>
<td>3</td>
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<td>3</td>
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<td>0</td>
<td>0</td>
<td>9</td>
<td>s_1</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>9</td>
<td>0</td>
<td>a_2</td>
</tr>
</tbody>
</table>

Here the next step becomes to make the top row reflect that \( a_1 \) and \( a_2 \) are basic variables.
5. (20 pts) A canning company is contracted to receive 60,000 pounds of ripe grade A tomatoes at 9 cents per pound from which it produces both canned tomato juice and tomato paste. They also will get 50,000 pounds of grade B tomatoes at 6 cents per pound. The canned products are packaged in 24 can cases. A can of juice requires 1 pound of fresh tomatoes, and a can of paste requires 1/3 pound only. Juice must use at least 60% grade A tomatoes and the paste must use at least 50% grade A tomatoes. The company’s share of the market is limited to 20,000 cans of juice and 60,000 cans of paste. The wholesale price per can of juice is $1 and for paste it is $0.50. The cost of processing a can of juice is $0.10 and the cost of processing a can of tomato paste is $0.20. Determine the optimum daily production mix. (Maximize the profit.) Again Set up Don’t Solve!

Solution: Start by defining some decision variables:

Let $A_J$ be the pounds of grade A tomatoes used in Juice.
Let $A_P$ be the pounds of grade A tomatoes used in Paste.
Let $B_J$ be the pounds of grade B tomatoes used in Juice.
Let $B_P$ be the pounds of grade B tomatoes used in Paste.

where each of the decision variables is non-negative.

Note we can determine the number of cans of Juice and cans of paste produced using:

Cans of Juice := $A_J + B_J \leq 20,000$ Market Share Juice
Cans of Paste := $3(A_P + B_P) \leq 60,000$ Market Share Paste

The company’s profit $z$ is defined by:

$$z = (1.00 - 0.10)(A_J + B_J) + (0.50 - 0.20)(3(A_P + B_P)) - 0.09(A_J + A_P) - 0.06(B_J + B_P)$$

In addition to the previously defined constraints we have:

$A_J + A_P \leq 60,000$ pounds of grade A tomatoes available
$B_J + B_P \leq 50,000$ pounds of grade B tomatoes available

The juice and the paste must satisfy the desired ratios of grade A tomatoes.

$$\frac{A_J}{A_J + B_J} \geq 0.60 \text{ and } \frac{A_P}{A_P + B_P} \geq 0.5$$
Putting it all together gives:

maximize \[ z = 0.9(A_J + B_J) + 0.3(A_P + B_P) - 0.09(A_J + A_P) - 0.06(B_J + B_P) \]
subject to
\[ A_J + B_J \leq 20,000 \]
\[ 3(A_P + B_P) \leq 60,000 \]
\[ A_J + A_P \leq 60,000 \]
\[ B_J + B_P \leq 50,000 \]
\[ 0.4A_J - 0.6B_J \geq 0 \]
\[ 0.5A_P - 0.5B_P \geq 0 \]
\[ A_J, A_P, B_J, B_P \geq 0 \]
545 Additional Problems

1. (10 pts) Show that for any two matrices $A$ and $B$, $(AB)^T = B^T A^T$.

Assume that $A$ is an $m \times n$ matrix and $B$ is an $n \times q$ matrix so that their product is defined. Thus, $AB$ is an $m \times q$ matrix, and its transpose is a $q \times m$ matrix. We can also note that $B^T A^T$ is the product $q \times n$ matrix and a $n \times m$ matrix resulting in again a $q \times m$. So each side of the equality is of the same size. We now only need to show that the arbitrary element in row $i$ and column $j$ of $(AB)^T$ is the same as the row $i$ column $j$ element in $B^T A^T$.

Denoting the entry in row $i$ column $j$ in a matrix $M$ as $M_{i,j}$ we can consider the following algebra:

$$ (AB)^T_{i,j} = (AB)_{j,i} = \sum_{k=1}^{n} A_{j,k} B_{k,i} = \sum_{k=1}^{n} A_{k,j}^T B_{i,k}^T = \sum_{k=1}^{n} B_{i,k}^T A_{k,j}^T = (B^T A^T)_{i,j} $$

which concludes the proof.

2. (10 pts) For a linear program written in standard form with constraints $A x = b$ and $x \geq 0$ show that $d$ is a direction of unboundedness if and only if $A d = 0$ and $d \geq 0$.

Solution:
First lets recall what a direction of unboundedness is. An $n$ by 1 vector $d$ is a direction of unboundedness if for all $x$ in $S$ (an LP’s feasible region), and $c \geq 0$ then

$$ x + c d \in S. $$

• ($\Rightarrow$) Assume here that $d$ is a direction of unboundedness, and that $c$ is a positive constant ($c = 0$ is trivial). Then we know that for any $x \in S$ that $x + c d$ is in $S$. This gives

$$ A(x + cd) = b \quad \Rightarrow \quad A x + c A d = b \quad \Rightarrow \quad b + c A d = b \quad \Rightarrow \quad c A d = 0 \quad \Rightarrow \quad A d = 0. $$

We also know that $x + c d \geq 0$ for all $x \geq 0 \in S$. Thus, $c d \geq 0$, and $d \geq 0$. 
(⇐) Assume that $Ad = 0$, and $d \geq 0$, and consider any point $x$ in the feasible region of our linear program. As $x$ is in the feasible region we know that $Ax = b$. Thus, for $c > 0$ consider the point $x + cd$

$$A(x + cd) = Ax + cAd = Ax + 0 = Ax = b.$$ 

Note we also need our new point “$x + cd$" to be non-negative which is true as $x \in S$, and $d \geq 0$. Thus, $x + cd \in S$, and $d$ is a direction of unboundedness.