Notes: Probabilistic Models in Operations Research

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Preface

These notes will serve as an introduction to the basics of solving probabilistic models in operations research. From the IUP course catalog the course will contain:

A survey of probabilistic methods for solving decision problems under uncertainty. Probability review, decision theory, queuing theory, inventory models, and Markov chains are covered. Uses technology to solve problems and interpret the results.

The majority of this course will follow the presentation given in the Operations Research: Applications and Algorithms text by Winston [7]. Relevant course material will start in chapter 12 of the text. I will supplement the Winston text with additional material from other popular books on operations research. For further reading you may wish to consult:

- Introduction to Operations Research by Hillier and Lieberman [2]
- Linear Programming and its Applications by Eiselt and Sandblom [1]
- Linear and Nonlinear Programming by Luenberger and Ye [3]
- Linear and Nonlinear Programming by Nash and Sofer [4]

My Apologies in advance for any typographical errors or mistakes that are present in this document. That said, I will do my very best to update and correct the document if I am made aware of these inaccuracies.

- John Chipell
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Chapter 1

Motivation

"Entire sections of them simply cannot be translated - the characters are legible and well-known, but when put together they do not say anything that leaves an imprint on the modern mind. "Like instructions for programming a VCR."

—Neal Stephenson, Snow Crash

The goal of many topics discussed in this course will be to make decisions in an uncertain environment. A brief course outline potentially includes:

- Review of Calculus and Probability
- Decision Making under Uncertainty
- Game Theory (Class Presentations)
- Markov Chains and Simulation
- Stochastic Processes
- Inventory Models

Let's start by looking at a basic State of the World Model and a motivating example.

1.1 State of the World Model

A simple model of decision making with uncertainty is given by:

Given a set of possible actions

\[ A = \{a_1, a_2, \ldots, a_n\} \]
an action is chosen. The state of the world $s_j$ is then observed with probability $p_j$, where

\[ s_j \in \{s_1, s_2, \ldots, s_m\}. \]

A reward $r_{ij}$ is given for choosing action $a_i$ and having the world in state $s_j$.

Considering this as a general model lets now look at a motivating example.

Example: ¹

Your selling newspapers at the corner of Philadelphia and Oakland, and each day you must determine how many newspapers you need to order.

- You may purchase newspapers for $0.20.
- You sell the newspapers for $0.25.
- Unsold newspapers are worthless!

On any given day you can sell between 6 and 10 papers with each possibility being equally likely. What action should you take?

Fitting the situation into the state of the world model:

\[ S = \{6, 7, 8, 9, 10\} \] is the demand for newspapers.

the probability of each of these world states is

\[ p_j = \frac{1}{5} \] for \( j \in \{6, 7, 8, 9, 10\} \]

and the actions set is

\[ A = \{6, 7, 8, 9, 10\} \] corresponding to newspapers purchased.

Assuming that $i$ papers are purchased and $j$ papers are in demand then the reward for each action is then given by:

\[ r_{ij} = 0.25i - 0.20i = 0.05i \quad \text{for } i \leq j \]

\[ r_{ij} = 0.25j - 0.20i \quad \text{for } i > j \]

where

\[ \min\{i, j\} \] papers are sold.

This leads to a reward matrix:

¹Slightly modified example form Winston Text
In the reward matrix stated. We didn’t consider purchasing less than 6 papers, and we didn’t consider purchasing more than 10, as these are the dominated actions.

**Dominated Action**: An action \( a_i \) is dominated by an action \( a_{i'} \), if

- for all \( s_j \in S \) the reward \( r_{ij} \leq r_{i'j} \)
- for some state \( s_j \) the reward \( r_{ij} < r_{i'j} \).

If \( a_i \) is a dominated action by \( a_{i'} \) then in no state of the world is \( a_i \) better than \( a_{i'} \) and in at least one instance \( a_i \) is inferior to \( a_{i'} \) giving us no reason to choose action \( a_i \).

We can see that the rows corresponding to ordering 5 papers is dominated by the action of choosing to order 6 papers. The action of choosing to order 11 papers is dominated by the action of ordering 10 papers.

**HOW DO WE CHOOSE AN ACTION TO TAKE?**

**Criterion for choosing an action**

There are several criterion that can be considered when choosing an action to take:

1. We could see what the lowest return is for each action, and choose the one that has the greatest or “best” worst outcome. This is the **Maximin Criterion**.
   - This leads to purchasing 6 papers.
   - You always make $0.30 dollars.
   - You never make more than $0.30 dollars.

2. You could look and see which action or opportunity has the best possible outcome. This is the **Maximax Criterion**.

<table>
<thead>
<tr>
<th>Papers Purchased ( i )</th>
<th>Papers Demanded ( j )</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>DA</td>
<td>5</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.1</td>
<td>0.35</td>
<td>0.35</td>
<td>0.35</td>
<td>0.35</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>-0.1</td>
<td>0.15</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>-0.3</td>
<td>-0.05</td>
<td>0.2</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>-0.5</td>
<td>-0.25</td>
<td>0</td>
<td>0.25</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>-0.7</td>
<td>-0.45</td>
<td>-0.2</td>
<td>0.05</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>-0.9</td>
<td>-0.65</td>
<td>-0.4</td>
<td>-0.15</td>
<td>0.1</td>
</tr>
</tbody>
</table>
• This leads to purchasing 10 papers.
• In the best case you make $0.50 dollars.
• You have the potential to lose $0.50 dollars as well.

3. The **Minimax Regret** criterion has a decision maker calculate the action \( a_i^* \) that maximizes the reward for each world state (the best action to pick if the world state is \( s_j \). Then calculate the regret or opportunity loss for action \( a_i \) in \( s_j \). That is

\[
r_{i,j} = r_{ij} = \text{the regret or opportunity loss}
\]

Here the action that minimizes the maximum of the regret matrix is chosen. For our problem the regret matrix is given by:

\[
r_{i,j} = 0.05 \times i \text{ selling all papers purchased.}
\]

<table>
<thead>
<tr>
<th>Papers Purchased</th>
<th>Papers Demanded</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( i \times j )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0.05</td>
<td>0.1</td>
<td>0.15</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.2</td>
<td>0</td>
<td>0.05</td>
<td>0.1</td>
<td>0.15</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.4</td>
<td>0.2</td>
<td>0</td>
<td>0.05</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.6</td>
<td>0.4</td>
<td>0.2</td>
<td>0</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.8</td>
<td>0.6</td>
<td>0.4</td>
<td>0.2</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Thus, the maximum regret for any given action is given by:

<table>
<thead>
<tr>
<th>Papers Ordered</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Regret</td>
<td>0.2</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Minimizing the regret gives yields the choice of ordering 6 or 7 papers.

4. Choosing the action that yields the largest expected reward is known as the **Expected Value Criterion**.

So how do we calculate the expected value of any of the given actions?

As we already know the reward for each world state and action combination we can use the given fact that each world state is equally likely to average the expected value of choosing any specific action. Thus,

<table>
<thead>
<tr>
<th>Papers Purchased</th>
<th>Expected Reward</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>( \frac{1}{5} (0.30 + 0.30 + 0.30 + 0.30 + 0.30) = $0.30 )</td>
</tr>
<tr>
<td>7</td>
<td>( \frac{1}{5} (0.10 + 0.35 + 0.35 + 0.35 + 0.35) = $0.30 )</td>
</tr>
<tr>
<td>8</td>
<td>( \frac{1}{5} (-0.10 + 0.15 + 0.40 + 0.40 + 0.40) = $0.25 )</td>
</tr>
<tr>
<td>9</td>
<td>( \frac{1}{5} (-0.30 - 0.05 + 0.20 + 0.45 + 0.45) = $0.15 )</td>
</tr>
<tr>
<td>10</td>
<td>( \frac{1}{5} (-0.50 - 0.25 + 0.00 + 0.25 + 0.50) = $0.00 )</td>
</tr>
</tbody>
</table>

This would lead to choosing to purchase either 6 or 7 papers.
Chapter 2

Introduction and Review

"Tread of lying in the sunshine, Staying home to watch the rain, You are young and life is long, and there is time to kill today."

—Pink Floyd

This section will serve as a basic review of the foundations of Calculus and Probability needed for the study of probabilistic operations research questions.

2.1 Integral Calculus:

We can recall the following integration rules form a calculus course:

\[
\begin{align*}
\int dx &= x + C \\
\int cf(x) \, dx &= c \int f(x) \, dx \\
\int f(x) + g(x) \, dx &= \int f(x) \, dx + \int g(x) \, dx \\
\int x^n \, dx &= \frac{x^{n+1}}{n+1} + C \\
\int \frac{1}{x} \, dx &= \ln(x) + C \\
\int e^{f(x)} f'(x) \, dx &= e^{f(x)} + C \\
\int e^x \, dx &= e^x + C \\
\int a^x \, dx &= \frac{a^x}{\ln(a)} + C \\
\int (f(x))^n f'(x) \, dx &= \frac{(f(x))^{n+1}}{n+1} + C \\
\int \frac{1}{f(x)} f'(x) \, dx &= \ln(f(x)) + C \\
\int a^{f(x)} f'(x) \, dx &= \frac{a^{f(x)}}{\ln(a)} + C
\end{align*}
\]

Here \( C \) is an arbitrary real constant, and \( n \neq -1 \), and \( a > 0, a \neq 1 \). When two functions are considered integration by parts gives:

\[
\int u(x)v'(x) \, dx = u(x)v(x) - \int v(x)u'(x) \, dx
\]
Recall that the concept of the integral is used to find the area under the curve $y = f(x)$ between two points $a$ and $b$ as

$$\lim_{h \to 0} \sum_{i=1}^{n} f(x_i)h = \int_{a}^{b} f(x) \, dx$$

where the interval from $a$ to $b$ is broken into $n$ subintervals each of size $h$.

It may also be useful to remind ourselves of the Fundamental Theorem of Calculus that gives for a continuous function $f(x)$ where $a \leq x \leq b$ then

$$\int_{a}^{b} f'(x) \, dx = f(x)\bigg|_{a}^{b} = f(b) - f(a).$$

Slightly more obscure is the Leibniz’s rule for differentiating an integral. In its general form (given a general parameter $\alpha$) yields:

$$\frac{\partial}{\partial \alpha} \int_{h(\alpha)}^{g(\alpha)} f(x, \alpha) \, dx = \frac{\partial h(\alpha)}{\partial \alpha} f(h(\alpha), \alpha) - \frac{\partial g(\alpha)}{\partial \alpha} f(g(\alpha), \alpha)) + \int_{g(\alpha)}^{h(\alpha)} \frac{\partial}{\partial \alpha} f(x, \alpha) \, dx$$

**Example**

Use Leibniz’s rule to evaluate the following:

$$\frac{\partial}{\partial y} \int_{0}^{y} yx^2 \, dx$$

$$\frac{\partial}{\partial y} \int_{0}^{y} yx^2 \, dx = \frac{4}{3} y^3.$$  

### 2.2 Probability Basics

It will be informative to be reminded of the basics of probability.

- **An Experiment** denotes any situation where the outcome is unknown.
- **The sample space** will denote all the possible outcomes of an experiment. For now we will denote the sample space by $S$.
- **An event**, $E$, is a collection of points in the sample space. Events are said to be **mutually exclusive** provided they have no points in common.

With each event in the sample space $E$ we denote its complement $\bar{E}$ as the points in the sample space that are not included in event $E$.

Following the Winston text [7] note that events must satisfy the rules of probability:
1. The probability of the event must be non-negative: \( P(E) \geq 0 \).

2. If \( E = S \) then the event contains all points in the sample space and \( P(E) = 1 \).

3. For \( E_1, E_2, \ldots, E_n \) mutually exclusive events then
   \[
   P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{k=1}^{n} P(E_k)
   \]

4. The probability of an events complement
   \[
   P(\bar{E}) = 1 - P(E).
   \]

**Conditional Probability**

For two events \( E_1 \) and \( E_2 \) we denote the probability of event \( E_2 \) occurs given event \( E_1 \) has occurred as
\[
P(E_2|E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}.
\]

Note that events \( E_1 \) and \( E_2 \) are independent if and only if
\[
P(E_2|E_1) = P(E_2) \text{ or equivalently } P(E_1|E_2) = P(E_1)
\]

**Examples**

Given a standard deck of 52 cards

- What is the probability that a card drawn is not a 5?
  Here we consider the probability of drawing a 5 and call this \( E_5 \). Then
  \[
P(E_5) = \frac{4}{52} = \frac{1}{13}
  \]
  The probability of not drawing a 5 is given by:
  \[
P(\bar{E}_5) = 1 - \frac{1}{13} = \frac{12}{13}
  \]

- Given that a black card is drawn what is the probability that it is a spade?
  Let’s define the following events:
  \[
  E_b = \text{a black card is drawn.} \implies P(E_b) = \frac{1}{2} \\
  E_s = \text{a spade is drawn.} \implies P(E_s) = \frac{1}{4}
  \]
Then the probability that a spade is drawn given the card drawn is black is

\[
P(E_s|E_b) = \frac{P(E_b \cap E_s)}{P(E_b)}
\]

so in order to complete the computation we need to know the probability of drawing a black spade (or \(P(E_b \cap E_s)\)) which has probability \(\frac{13}{52} = \frac{1}{4}\). Thus,

\[
P(E_s|E_b) = \frac{P(E_b \cap E_s)}{P(E_b)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}
\]

Note this make sense as half the black cards are spades.

- Are the events of \(E_b\) and \(E_s\) independent?
  Here we need to see that
  \[
P(E_s|E_b) = P(E_s)
  \]
  in order for the events to be independent. This is not the case using the probabilities found above as \(P(E_s) = \frac{1}{4}\) and \(P(E_s|E_b) = \frac{1}{2}\).

Assume that we have two fair dice.

- What is the probability that the two dice will add up to 7 or 11? Assume that each die has only six faces.
  Let \(E_{i,j}\) denote rolling a value of \(i\) on die \(j\) for \(i \in \{1, 2, \ldots, 6\}\) and \(j \in \{1, 2\}\). Then

\[
P(E_{i,j}) = \frac{1}{6} \quad \forall \ i, j \text{ pairs.}
\]

We can also denote the event that the two dice sum to a given value be denoted by \(E_{sk}\) where \(k\) is an integer.

Let’s now consider the way we can arrive at a sum of 7 or 11:

<table>
<thead>
<tr>
<th>First die</th>
<th>Second die</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>2 or 6</td>
<td>7 or 11</td>
</tr>
<tr>
<td>6</td>
<td>1 or 5</td>
<td>7 or 11</td>
</tr>
</tbody>
</table>

We can now sum the probability of each row of the table occurring to obtain the probability of rolling two dice that have a sum of 7 or 11.

\[
P(E_{s7} \cup E_{s11}) = 4 \left( \left( \frac{1}{6} \right) \left( \frac{1}{6} \right) \right) + 2 \left( \left( \frac{1}{6} \right) \left( \frac{1}{3} \right) \right)
\]

\[
= \frac{2}{9}
\]
• Given that the total of the two dice is 5 what is the probability that the first die rolled was a 2?

Using our previously defined notation we want to find \( P(E_{2,1}|E_{s5}) \).

\[
P(E_{2,1}|E_{s5}) = \frac{P(E_{2,1} \cap E_{s5})}{P(E_{s5})}
\]

In order to get a feel for the \( P(E_{s5}) \) we can make the following table:

<table>
<thead>
<tr>
<th>First die</th>
<th>Second die</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

Summing the probability of each row of the table gives the probability of the two dice values add to 5. Thus,

\[
P(E_{s5}) = 4 \left( \left( \frac{1}{6} \right) \left( \frac{1}{6} \right) \right) = \frac{1}{9}
\]

The probability that the first dice rolled had value 2 and the sum of the dice is 5 is

\[
P(E_{2,1} \cap E_{s5}) = \left( \frac{1}{6} \right) \left( \frac{1}{6} \right) = \frac{1}{36}.
\]

Thus,

\[
P(E_{2,1}|E_{s5}) = \frac{P(E_{2,1} \cap E_{s5})}{P(E_{s5})}
\]

\[
= \frac{1}{36} \times \frac{1}{9}
\]

\[
= \frac{1}{9}
\]

Note this can be seen in the table as well as there is only one of the four cases that sum to 5 in which the value on the first die is 2.

### 2.2.1 Bayes’ Rule

Assume that there are \( n \) mutually exclusive states that make up the set of all possibilities and are thus collectively exhaustive before an experiment is observed. Giving each state a label \( S_i \) and assuming that a decision maker has assigned to each state a probability \( P(S_i) \) these will be the prior probabilities. We can assume that the outcome of the experiment \( O_j \) is in some way based on the state of the world. Thus,

\[
P(O_j|S_i) = \text{is the likelihood of outcome } O_j \text{ given } S_i
\]
Knowing the likelihood of $O_j$ given world state $S_i$ we can now derive an expression for the probability of world state $S_i$ given we observe outcome $O_j$ from the experiment.

$$P(S_i|O_j) = \frac{P(S_i \cap O_j)}{P(O_j)} \implies P(S_i \cap O_j) = P(S_i|O_j)P(O_j)$$

The probability that the experiment has an observed outcome $O_j$ is

$$P(O_j) = P(O_j \cap S_1) + P(O_j \cap S_2) + \cdots + P(O_j \cap S_n)$$

$$= \sum_{i=1}^{n} P(O_j \cap S_i)$$

$$= \sum_{i=1}^{n} P(O_j|S_i)P(S_i).$$

Making appropriate substitutions, yields Bayes’ rule

$$P(S_i|O_j) = \frac{P(S_i \cap O_j)}{P(O_j)} = \frac{P(O_j|S_i)P(S_i)}{\sum_{i=1}^{n} P(O_j|S_i)P(S_i)}$$

**Example**

From the Winston Text [7]: A desk contains three drawers. Drawer 1 contains two gold coins. Drawer 2 contains one gold coin and one silver coin. Drawer 3 contains two silver coins. I randomly choose a drawer and then randomly choose a coin. If a silver coin is chosen, what is the probability that I chose drawer 3?

Let

$S_i$ be the state of choosing each drawer $i \in \{1, 2, 3\}$.

$O_j$ be the outcome of selecting a coin $j \in \{g, s\}$

where $g$ denotes gold and $s$ denotes silver. We aim to find

$$P(S_3|O_s) = \text{Selecting drawer 3 given we chose a silver coin.}$$

From Bayes’ rule we know:

$$P(S_3|O_s) = \frac{P(O_s|S_3)P(S_3)}{P(O_s|S_1)P(S_1) + P(O_s|S_2)P(S_2) + P(O_s|S_3)P(S_3)}$$

Thus we need:

$$P(O_s|S_1) = \frac{0}{2}, \quad P(S_1) = \frac{1}{3}$$

$$P(O_s|S_2) = \frac{1}{2}, \quad P(S_2) = \frac{1}{3}$$

$$P(O_s|S_3) = \frac{2}{2}, \quad P(S_3) = \frac{1}{3}$$
Then
\[ P(S_3|O_s) = \frac{\frac{1}{3}}{0 + \frac{1}{6} + \frac{1}{3}} = \frac{2}{3} \]

The probability that drawer 3 was chosen given a silver coin was chosen is \( \frac{2}{3} \).

### 2.2.2 Random Variables and Distributions

**Random variables** (often denoted by \( X \)) are a way of associating a number with each point in an experiment’s sample space. They may be discrete or continuous.

**Examples:**

- **Discrete:**
  - Rolling a dice
  - Flipping a coin

- **Continuous:**
  - Time to failure (light bulb burning out).

Each random variable is quantified by a **probability density function** (pdf)\(^1\) such that the output of the pdf is non-negative, and the sum of the outputs of the pdf is one over the domain. Thus for \( d(x) \) and \( f(x) \) discrete and continuous pdf functions respectively we have:

<table>
<thead>
<tr>
<th>function</th>
<th>Domain</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrete: ( d(x) )</td>
<td>( x = a, a + 1, \ldots, b )</td>
<td>( d(x) \geq 0, \sum_{x=a}^{b} d(x) = 1 )</td>
</tr>
<tr>
<td>Continuous: ( f(x) )</td>
<td>( a \leq x \leq b )</td>
<td>( f(x) \geq 0, \int_{a}^{b} f(x) , dx = 1 )</td>
</tr>
</tbody>
</table>

The **cumulative distribution function** or cdf is defined as:

\[
p(x \leq X) = \left\{ \begin{array}{ll}
D(x) = \sum_{x=a}^{X} d(x), & \text{for Discrete } d(x) \\
F(x) = \int_{a}^{X} f(x) \, dx, & \text{for Continuous } f(x).
\end{array} \right.
\]

where \( d(x) \) and \( f(x) \) are the above defined probability density functions.

---

\(^1\)a pdf is sometimes called a probability mass function for discrete variables
Examples:

- What would the cdf look like for rolling a dice or drawing a given rank card form a standard deck 52 cards?
- Given a random variable $X$ having a density function
  \[
  f(x) = \begin{cases} 
  3x & \text{if } 0 \leq x \leq \sqrt{2} / 3 \\
  0 & \text{otherwise},
  \end{cases}
  \]
  find the probability $P(\frac{1}{2} \leq X \leq \frac{3}{4})$.

  Here the answer is found by calculating the integral:
  \[
  P\left(\frac{1}{2} \leq X \leq \frac{3}{4}\right) = \int_{\frac{1}{2}}^{\frac{3}{4}} 3x \, dx = \frac{3}{2} x^2 \bigg|_{\frac{1}{2}}^{\frac{3}{4}} = \frac{3}{2} \left(\left(\frac{3}{4}\right)^2 - \left(\frac{1}{2}\right)^2\right) = \frac{5}{16}
  \]

2.2.3 Mean and Variance

The mean of a random variable is its expected value. The mean tells us the location of a random variables center of mass. Denoting the mean of a random variable $E(X)$ it is computed by
\[
E(X) = \sum_{i} x_i P(X = x_i)
\]
for discrete random variable $X$, and as
\[
E(X) = \int_{-\infty}^{\infty} x f(x) \, dx
\]
for a continuous random variable. Note the expected value is just summing over all possible values of a random variable weighted the probability that that value occurs. Note we can find the expected value of functions of random variables in the same fashion as:
\[
E(h(X)) = \sum_{i} h(x_i) P(X = x_i)
\]
for discrete random variable $X$, and as
\[
E(h(X)) = \int_{-\infty}^{\infty} h(x) f(x) \, dx
\]
The variance of a random variable measures the dispersion of the random variable, $X$, about its expected value. Thus,
\[
\text{var}X = E((X - E(X))^2) = \sum_{i} (x_i - E(X))^2 P(X = x_i) \text{ for discrete cases,}
\]
and
\[
\text{var}X = E((X - E(X))^2) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x) \, dx \text{ for continuous cases.}
\]

For homework it can be shown that show that:
\[
\text{var}X = E((X - E(X))^2) = E(X^2) - E(X)^2.
\]

For any random variable we denote the standard deviation \( \sigma_X \) as
\[
\sigma_X = \sqrt{\text{var}X}.
\]

Example:

Consider the exponential distribution
\[
f(x) = \begin{cases} 
\lambda e^{-\lambda x} & \text{if } x \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]
for the random variable \( X \) where \( \lambda > 0 \) is a fixed parameter.

The cumulative distribution function is:
\[
F(x) = -e^{-\lambda x} + 1
\]
and Figure 2.2.1 plots of \( f(x) \) and \( F(x) \) when the model parameter \( \lambda = 1 \).

Calculating the mean (expected value) of the exponential distribution gives:
\[
E(X) = \int_{-\infty}^{\infty} xf(x) \, dx = \int_{0}^{\infty} x\lambda e^{-\lambda x} \, dx
\]

Figure 2.2.1: Plots of the exponential distribution (left), and its cumulative distribution function (right) with \( \lambda = 1 \).
Here integration by parts is used

\[ \int u \, dv = uv - \int v \, du \]

so we choose

\[ u = x \quad \text{and} \quad dv = \lambda e^{-\lambda x} \rightarrow v = -e^{-\lambda x} \quad \text{and} \quad du = dx. \]

Thus,

\[
E(X) = \int_0^\infty x \lambda e^{-\lambda x} \, dx \\
= -xe^{-\lambda x}\bigg|_0^\infty + \int_0^\infty e^{-\lambda x} \, dx \\
= 0 - \frac{1}{\lambda}e^{-\lambda x}\bigg|_0^\infty \\
= 0 + \frac{1}{\lambda}e^{-\lambda(0)} \\
= \frac{1}{\lambda}
\]

The mean of the exponential distribution is then \( \frac{1}{\lambda} \).

For the variance we look to calculate

\[ \text{var} X = E(X^2) - E(X)^2 \]

so we compute

\[
E(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x} \, dx
\]

using integration by parts first. Here we choose

\[ u = x^2 \quad \text{and} \quad dv = \lambda e^{-\lambda x} \, dx \quad \Rightarrow \quad du = 2x \, dx \quad \text{and} \quad v = -e^{-\lambda x} \]

giving

\[
E(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x} \, dx \\
= -x^2e^{-\lambda x}\bigg|_0^\infty + \int_0^\infty 2xe^{-\lambda x} \, dx \\
= 0 + \int_0^\infty 2xe^{-\lambda x} \, dx \\
= 2\frac{1}{\lambda}E(X).
\]

Putting it all together we can obtain the variance of the exponential distribution.

\[
\text{var} X = E(X^2) - E(X)^2 = \frac{2}{\lambda}E(X) - E(X)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}
\]
2.2.4 The Normal Distribution

The normal distribution will be potentially useful during this course. Thus, for a continuously random variable $X$ normally distributed with expected value $\mu$ (the mean), and standard deviation $\sigma > 0$ (the square root of the variance). The probability density function $f(x)$ is given by:

$$f(x) = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right). \quad (2.2.1)$$

The following graphic shows a plot of the normal distribution with $\mu = 0.3$ and $\sigma = 0$. See the link for an interactive link for the plot.

http://www.math.iup.edu/~jchrispe/MATH446_546/NormalPDFPlot.html

The following are important properties for the normally distributed random variables.

- If $X$ is a normally distributed variable with mean $\mu$ and variance $\sigma^2$, written as $X$ is $N(\mu, \sigma^2)$, then $cX$ is $N(c\mu, c^2\sigma^2)$.

- If $X$ is $N(\mu, \sigma^2)$, then $X + c$ is $N(\mu + c, \sigma^2)$.

- If $X_1$ is $N(\mu_1, \sigma_1^2)$ and $X_2$ is $N(\mu_2, \sigma_2^2)$ and $X_1$ and $X_2$ are independent, then $X_1 + X_2$ is $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. 


The Gaussian Distribution

Note we can standardize any normal distribution using a transformation using the properties listed above. Let's consider \( Z \) is a normal random variable with mean \( \mu = 0 \) and variance \( \sigma^2 = 1 \). This is called the Gaussian distribution. Let's also assume we have a mechanism in place for calculating probabilities using the Gaussian distribution. SAGE will nicely compute probabilities using this distribution:

```python
sigma = 1;
Z = RealDistribution('gaussian', sigma);
```

Then note that

```python
print Z.distribution_function(0);
print Z.cum_distribution_function(0);
```

yields results of

\[ 0.398942280401 \quad \text{and} \quad 0.5 \]

as expected. Thus, if we want to solve for the probability that a random variable \( X \) with normal distribution having mean \( \mu = 1 \) and variance \( \sigma^2 = 0.01 \). Then

\[
P(X \leq 0.97) = P \left( Z \leq \frac{0.97 - \mu}{\sigma} \right) = P \left( Z \leq \frac{0.97 - 1}{0.1} \right) = P(Z \leq -0.3)
\]

and using SAGE

```python
print Z.cum_distribution_function(-0.3)
```

yields 0.382088577811. This value could also be found using the Table on page 724 of the Winston text [7].

Example

The weekly demand for ice (in 25 pound bags) at the Turtle Pit convenience store is normally distributed with \( \mu = 45 \) and standard deviation \( \sigma = 6 \). How many bags of ice must be in stock at the beginning of a week if the Turtle Pit is to have only a 5% chance of running out of ice by the end of the week?

Here we let \( X \) be a random variable with mean \( \mu = 45 \) and variance \( \sigma^2 = 36 \). We want to know the value \( x \) such that \( P(X \leq x) = 0.95 \). We could set up the integral by hand or use SAGE.

```python
sigma = 1;
Z = RealDistribution('gaussian', sigma);
z = Z.cum_distribution_function_inv(0.95);
```
Gives a value of $z$ that is related to $x$ as

$$z = \frac{x - \mu}{\sigma} \implies x = \sigma z + \mu$$

and therefore

$$x = 6(1.64485362695) + 45 = 54.8691217617$$

and the Turtle Pit should have at least 55 bags of ice on hand at the start of the week if they are to only have a 5% chance of running out by the end of the week.

Note this problem could have been solved also by setting the standard deviation $\text{sigma} = 6$ in sage and then adding on the mean to the returned answer.

More information on using probability distributions in SAGE can be found at:

Example:

From the Winston text [7] page 729. Before it burns out a light bulb gives $X$ hours of light, where $X$ is normally distributed with $\mu = 500$, and variance $\sigma^2 = 400$. If we have three light bulbs what is the probability that they will give a total of at least 1460 hours of light?

Let $X_i$ be the $N(500, 400)$ random variable for each of the light bulbs. Note each random variable is independent. So we can create a random variable

$$X_s = X_1 + X_2 + X_3$$

with

$$\mu_s = 3\mu = 1500 \text{ and } \sigma_s^2 = 3\sigma^2 = 1200$$

$$P(X_s > 1460) = 1 - P(X_s \leq 1460)$$

$$\text{sigma} = 1.0;$$
$$Xs = \text{RealDistribution('gaussian', sigma);}$$
$$p = 1 - Xs.\text{cum_distribution}\_\text{function}((1460 - 1500)/1200**0.5);$$

Using the above SAGE commands or the distribution table to find the answer gives:

$$P(X_s > 1460) = 0.875893460505038 \text{ or } 0.8749 \text{ using table.}$$
Chapter 3

Utility Theory

"I've done the math enough to know the dangers of our second guessing. Deemed to crumble unless we grow, and strengthen our communication."

- Maynard Keynes

This section will concern the Von Neumann-Morgenstern utility function and how it may be used to aid when making decisions with uncertainty.

Let's begin with a definition: A situation where an individual can receive, a reward $r_i$ with probability $p_i$ for $i = 1, \ldots, n$ is a lottery. We denote this with:

$$\text{Lottery}(p_1, r_1; p_2, r_2; \ldots; p_n, r_n)$$

Lotteries are often represented as trees. Here the branches denote the possible outcomes.

Example: Given the lottery $(\frac{1}{6}, $900; $\frac{5}{6}, $0)$ we could write:

$$L \begin{array}{c}
\frac{1}{6} \quad $900 \\
\frac{5}{6} \quad $0
\end{array}$$

We can use lotteries to compare situations.

Consider the following two lotteries:

$$L_1 \begin{array}{c}
1 \quad $2000
\end{array} \quad \text{and} \quad L_2 \begin{array}{c}
\frac{1}{2} \quad $5000 \\
\frac{1}{2} \quad $0
\end{array}$$
Here in the first lottery (Lottery $L_1$) with perfect certainty we gain $2000. In lottery $L_2$ we are flipping a coin and have a 50/50 chance of nothing or $5000.

Looking at the expected return of each gives:

$$
\text{Expected Return } L_1 = $2000
$$

$$
\text{Expected Return } L_2 = $2500
$$

The expected return of $L_2$ is larger; however, most people if given the chance will pick $L_1$ over $L_2$ as it has less risk or uncertainty.

Assuming that you must make a choice we write:

$$
L_1 p L_2 \text{ meaning } L_1 \text{ is prefered over } L_2,
$$

$$
L_1 i L_2 \text{ means } L_1 \text{ and } L_2 \text{ are equivalent.}
$$

In the second case the $i$ can be thought of as signifying indifference.

The goal now becomes to find a method we can use to decide the best or preferred choice between several different lotteries.

**Example Problem:** Given the following four lotteries come up with a way of ranking the lotteries in order of preference.

$$
L_1 \quad \begin{array}{c}
1 \\
\frac{1}{2}
\end{array} \quad $20,000
$$

$$
L_2 \quad \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array} \quad $60,000 \quad \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array} \quad $0
$$

$$
L_3 \quad \begin{array}{c}
1 \\
\frac{95}{100}
\end{array} \quad $30,000 \quad \begin{array}{c}
\frac{5}{100}
\end{array} \quad -$15,000
$$

Note that you must make a choice!

To rank the four lotteries we will use the Von Neumann - Morgenstern Approach. Here we first identify the most and least favorable outcomes.

$$
\text{Most := $60,000} \quad \text{Least := -$15,000}
$$
All other possible outcomes are placed in the set

\[ O = \{0, 20000, 30000\} \]

and for each of the rewards \( r_i \in O \) we determine the probability \( p_i \) such that the lottery

\[ L_{r_i} \]

\[ 1 \]

\[ r_i \]

and

\[ L_{bw} \]

\[ p_i \]

\[ 0.5 \]

\[ $60,000 \]

\[ (1 - p_i) \]

\[ -$15,000 \]

are such that we are indifferent between them. That is \( L_{r_i} \sim L_{bw} \).

For our example consider:

\[ L_{bw} \]

\[ 0.5 \]

\[ $60,000 \]

\[ 0.5 \]

\[ -$15,000 \]

\[ (*) \]

\[ L_{bw} \]

\[ 0.8 \]

\[ $60,000 \]

\[ 0.2 \]

\[ -$15,000 \]

\[ (**) \]

\[ L_{bw} \]

\[ 0.95 \]

\[ $60,000 \]

\[ 0.05 \]

\[ -$15,000 \]

\[ (***) \]

These lotteries can be used to construct other lotteries that only involve the best and the worst outcome, such that we are indifferent between the created lottery and one of the original 4.

Consider

\[ L_1 \]

\[ 1 \]

\[ $20,000 \]

From \( (**) \) we see that we are indifferent between \( L_1 \) and the lottery
For the lottery

\[ L_1 \left\langle \begin{array}{c} 0.8 \quad $60,000 \\ 0.2 \quad -$15,000 \end{array} \right. \]

use (*) above to create a compound lottery, or a lottery where the reward of one of the outcomes it to play another lottery. A simple lottery is any lottery that is not complex. Here \( L_2 \) may be written and then reduced in the following manner:

\[ L_2 \left\langle \begin{array}{c} \frac{1}{2} \quad $60,000 \\ \frac{1}{2} \quad 0 \end{array} \right. \]

\[ L_2' \left\langle \begin{array}{c} 0.5 \quad $60,000 \\ 0.5 \quad 0 \end{array} \right. \]

Which reduces to the simple lottery

\[ L_2' \left\langle \begin{array}{c} 0.5 + 0.25 - $60,000 \\ (0.5)(0.5) - -$15,000 \end{array} \right. \]

\[ L_2' \left\langle \begin{array}{c} 0.75 \quad $60,000 \\ 0.25 \quad -$15,000 \end{array} \right. \]

We make note that we now have \( L_2' \times L_2 \).

For \( L_3 \) we make use of (*) and see that

\[ L_3 \left\langle 1 \quad $0 \right. \]

is equivalent to the lottery

\[ 0.8 \quad $60,000 \]
\[ 0.2 \quad -$15,000 \]
Lastly we consider the lottery

\[
L_4 = \begin{cases} 
\frac{95}{100} & \text{\$30,000} \\
\frac{5}{100} & \text{\$15,000}
\end{cases}
\]

Using (***), we can find the lottery

\[
L'_4 = \begin{cases} 
0.9025 & \text{\$60,000} \\
0.0975 & \text{\$15,000}
\end{cases}
\]

such that \( L'_4 L_4 \). Since we are indifferent to each lottery

\[ L'_i L_i \forall i \in \{1, 2, 3, 4\} \]

we can rank the original lotteries by comparing the lotteries \( L'_1, L'_2, L'_3, \) and \( L'_4 \). Here the preference is given to the lottery with the greatest chance of having the “Most Favorable” outcome.

To summarize our rankings we have:

\[ L_4 p L_1 p L_2 p L_3 \]

The utility of the reward \( r_i \) is given by \( q_i \) where

\[ u(r_i) = q_i \]

and is found by finding the value of \( q_i \) such that a decision maker is indifferent between certain reward \( r_i \) and the lottery involving only the most favorable and least favorable outcomes:

\[
\begin{cases} 
q_i & \text{Most Fav.} \\
(1 - q_i) & \text{Least Fav.}
\end{cases}
\]
We note that this gives

\[ u(\text{Most Favorable}) = 1, \text{ and } u(\text{Least Favorable}) = 0. \]

For our example problem the utility function is found to have the following points:

\[
\begin{align*}
  u(60000) &= 1 \\
  u(30000) &= 0.95 \\
  u(20000) &= 0.8 \\
  u(0) &= 0.5 \\
  u(-15000) &= 0
\end{align*}
\]

The expected utility for a lottery is given by (using Winstons notation)

\[
E(U \text{ for } L) = \sum_{i=1}^{n} p_i u(r_i)
\]

For each of the different lotteries we had in our example:

\[
\begin{align*}
  E(U \text{ for } L_1) &= 1u(20000) = 0.8 \\
  E(U \text{ for } L_2) &= 0.5u(60000) + 0.5u(0) = 0.5(1) + 0.5(0.5) = 0.75 \\
  E(U \text{ for } L_3) &= 1u(0) = 0.5 \\
  E(U \text{ for } L_4) &= 0.95u(30000) + 0.05u(-15000) = 0.95(0.95) + 0 = 0.9025
\end{align*}
\]

Ranking the expected value of the lotteries is yields the same preferences we have already seen:

\[ L_4 p L_1 p L_2 p L_3 \]

The Expected Value Criterion:

We can use the expected value then to denote our preference between two lotteries \( L_1 \) and \( L_2 \) as follows:

- \( L_1 p L_2 \) if and only if \( E(U \text{ for } L_1) > E(U \text{ for } L_2) \)
- \( L_1 i L_2 \) if and only if \( E(U \text{ for } L_1) = E(U \text{ for } L_2) \)

It should be noted that in order to use the above expected value criterion to rank the different lotteries given to us in the example we have just worked that some assumptions have been made. These are the Von Neumann - Morgenstern Axioms. Provided a decision makers preferences meet the following criterion then he or she can rank lotteries use the for mentioned process.

Complete Ordering For any two rewards \( r_1 \) and \( r_2 \) either one is preferred to the other or the decision maker is indifferent between \( r_1 \) and \( r_2 \). Note we have a transitive property where

\[
\text{if } r_1 \mathrel{p} r_2 \text{ and } r_2 \mathrel{p} r_3 \Rightarrow r_1 \mathrel{p} r_3
\]
Continuity If a decision maker prefers \( r_1 \) to \( r_2 \) and \( r_2 \) to \( r_3 \) then for some \( c \in (0, 1) \) a decision maker will be indifferent between a lottery \( L_1 \) with certain outcome \( r_2 \) and the lottery

\[
\begin{array}{c}
L_2 \\
\downarrow c \quad \downarrow r_1 \\
(1 - c) \quad r_3
\end{array}
\]

Independence Assuming a decision maker is indifferent between \( r_1 \) and \( r_2 \), then for some other reward \( r_3 \) a decision maker will be indifferent between lotteries \( L_1 \) and \( L_2 \) where

\[
\begin{array}{c}
L_1 \\
\downarrow c \quad \downarrow r_1 \\
(1 - c) \quad r_3
\end{array} \quad \text{and} \quad \begin{array}{c}
L_2 \\
\downarrow c \quad \downarrow r_2 \\
(1 - c) \quad r_3
\end{array}
\]

for all values of \( c \in (0, 1) \).

Unequal Probability If a decision maker prefers \( r_1 \) to \( r_2 \) and has the option of the lotteries:

\[
\begin{array}{c}
L_1 \\
\downarrow p_1 \quad \downarrow r_1 \\
(1 - p_1) \quad r_2
\end{array} \quad \text{and} \quad \begin{array}{c}
L_2 \\
\downarrow p_2 \quad \downarrow r_1 \\
(1 - p_2) \quad r_2
\end{array}
\]

The lottery with the maximum probability

\[\max\{p_1, p_2\}\]

will be preferred.

Compound Lottery Axiom Given a compound lottery \( L \) with \( i = 1, \ldots, n \) states yielding reward \( r_i \) with probability \( p_i \) for each state. Then \( L \) is equal to the simple lottery \( L' \) where

\[L' = (p_1, r_1; p_2, r_2; \ldots; p_n, r_n).\]
Utility Function

In order to rank the different lotteries we created a few discrete points on a utility function. In general we can find a utility function by assuming that the most favorable outcome has utility one and the least has utility zero and asking questions that involve:

\[
\begin{align*}
\text{Most Favorable} & := $60,000 \quad \text{and} \quad \text{Least Favorable} := -$15,000 \\
\end{align*}
\]

and finding the value of the reward \( x_{1/2} \) that makes a decision maker indifferent between a lottery with certain reward \( x_{1/2} \) and the lottery

\[
\begin{align*}
0.5 & \quad $60,000 \\
0.5 & \quad -$15,000
\end{align*}
\]

Then,

\[
u(x_{1/2}) = 0.5u(60,000) + 0.5u(-15,000) = 0.5(1) + 0.5(0) = 0.5
\]

Maybe in this instance

\[
x_{1/2} = 0
\]

as it was determined to be in the earlier example.

Let's ask the question again with a different maximum outcome. This time find the value \( x_{1/4} \) such that the decision maker is indifferent between a lottery with a certain outcome of \( x_{1/4} \) and the lottery

\[
\begin{align*}
0.5 & \quad $20,000 \\
0.5 & \quad -$15,000
\end{align*}
\]

Here we can use what we already have determined to help.

\[
u(x_{1/4}) = 0.5u(20,000) + 0.5u(-15,000) = 0.5(0.8) + 0.5(0) = 0.4
\]

and \( x_{1/4} = -$4,000 \), as here we view the given lottery the same as throwing that sum of money away.

Hopefully at the end of the day the curve we create is smooth. Plotting all of the values we have for our decision maker's utility function.
A decision maker’s utility shows us their attitude toward risk. Making the following definitions:

- **certainty equivalent**: the payoff that makes a decision maker indifferent between the given payoff amount and a given lottery $L$. Denoted as $CE(L)$

- **risk premium**: For a given lottery this will be the expected value of the lottery $L$ minus the certainty equivalent. Denoting the risk premium as $RP(L)$

$$RP(L) = EV(L) - CE(L)$$

- **degenerate lottery**: Any lottery where only one outcome can occur.

We can now classify non-degenerate lotteries as:

- risk-averse if and only if $RP(L) > 0$.
- risk-neutral if and only if $RP(L) = 0$.
- risk-seeking if and only if $RP(L) < 0$.

**Example**

Consider again the lottery:
The expected value of this lottery is

$$EV(L_4) = 0.05(-15,000) + 0.95(30,000) = $27,750$$

The expected utility of lottery $L_4$ using our predefined utility function is:

$$E(U \text{ for } L_4) = 0.95u(30000) + 0.05u(-15000) = 0.95(0.95) + 0 = 0.9025$$

We find the certainty equivalent of lottery $L_4$ by finding the reward value $r = CE(L_4)$ such that

$$u(r) = E(U \text{ for } L_4) \implies u(r) = 0.9025$$

using our utility function $r = $26,800, and the risk premium for lottery $L_4$ is

$$RP(L_4) = EV(L_4) - CE(L_4) = 27,750 - 26,800 = $950.$$ showing that we are risk averse.

http://www.math.iup.edu/~jchrispe/MATH446_546/RiskPremiumIllustration.html
Note that the decision makers tendency for risk can be seen in the shape of their utility function. Here

- A risk-averse person will have a utility function that is “strictly concave” (concave down).
- A risk-neutral decision maker will have a utility function that is “linear”.
- A risk-seeking decision maker will have a utility function that is “convex” (concave up).

**Example**

Show that a person having a utility function of \( u(x) = x^{1/2} \) has a risk averse attitude toward decision making.

Here we can use the second derivative test. Then

\[
u'(x) = \frac{1}{2}x^{-\frac{1}{2}}\]

and

\[
u''(x) = -\frac{1}{4}x^{-\frac{3}{2}} < 0, \quad \forall \ x \in (0, \infty).
\]

Showing that the decision maker has a risk averse attitude.

**Example**

Given a decision maker has a utility function for monetary gains \( x \) given by \( u(x) = (x + 10,000)^{1/2} \).

- Show that the decision maker is indifferent between the status quo, and the lottery:

\[
L \quad \frac{1}{3} \quad \frac{2}{3} \quad \text{$80,000$} \quad \text{-$10,000$}
\]

- If there is a 10% chance that a painting that is valued at $10,000 will be stolen in the next year, what is the most (per year) that the decision maker will be willing to pay for insuring the painting against loss?

Note these are questions taken form the end of section 13.2 in the Winston text. For the first question we aim to see that the expected utility of the lottery \( L \) has the same expected
utility as $0$. Thus,

\[
E(U_{forL}) = \frac{1}{3}u(80000) + \frac{2}{3}u(-10000)
\]

\[
= \frac{1}{3}(80000 + 10000)^{1/2} + \frac{2}{3}(-10000 + 10000)^{1/2}
\]

\[
= \frac{1}{3}(90000)^{1/2}
\]

\[
= \frac{300}{3}
\]

\[
= \$100
\]

We compare this with $u(0)$ a lottery with certain outcome of zero.

\[
u(0) = (0 + 10000)^{1/2} = $100.
\]

Showing that the decision maker is indeed indifferent between the given lottery and the status quo.

For the second problem we consider this as the comparison between to lotteries:

1. A lottery $L_1$ with certain outcome $x$, where $x$ denotes the amount we should pay for the insurance.

2. The lottery $L_2$ where we have

\[
L_2 \overset{0.9}{\longrightarrow} \$0 \overset{0.1}{\longrightarrow} -$10,000
\]

and we wish to be indifferent between the two lotteries:

\[
E(U_{ofL_2}) = 0.9u(0) + 0.1u(-10000)
\]

\[
= 0.9(0 + 10000)^{1/2} + 0.1(-10000 + 10000)^{1/2}
\]

\[
= 0.9(10000)^{1/2}
\]

\[
= 90
\]

We now look to find the value $x$ such that $u(x) = 90$. Thus,

\[
90 = (x + 10000)^{1/2}
\]

\[
90^2 - 10000 = x \implies x = -$1,900
\]

and the decision maker would be willing to pay at most $1900 for the paintings insurance.
A Common Utility Function

One common utility function used is

\[ u(x) = 1 - e^{-x/R} \]

where \( R > 0 \) is an adjustable parameter known as the risk tolerance. One way to calculate the value of \( R \) is to consider adjusting it until the decision maker is indifferent between a lottery with no pay off at all, and a lottery in which there is an equal probability between winning \( R \) dollars and losing \( R/2 \) dollars.

Example:

From the Winston Text: My current income is $40,000. I believe that I owe $8000 in taxes. For $500 I can hire a CPA to review my tax return; there is a 20% chance that she will save me $4000 in taxes. My utility function of my disposable is given by \( \sqrt{x} \) where \( x \) is my disposable income. Should I hire the CPA? Assume that my disposable income is what I have left after I pay my taxes and the accountant.

Set this up as the comparison of two lotteries:

In the first lottery the disposable income is $40,000 - $8,000 = $32,000.

\[ L_1 \quad 1 \quad 1 \quad 32,000 \]

and the second lottery with two different disposable income values:

\[ L_2 \quad 0.8 \quad 31,500 \quad 0.2 \quad 35,500 \]

80% chance of $40,000 - $8,000 - $500 = $31,500
20% chance of $40,000 - $4,000 - $500 = $35,500

Here the expected utility of each lottery is given by:

\[ E(U \ for \ L_1) = 1u(32,000) = 32,000^{1/2} = 178.885438 \]
\[ E(U \ for \ L_2) = 0.8u(31,500) + 0.2u(35,500) = 179.6688021 \]

So we should choose to pay the accountant.
3.1 Flaws in Utility Theory

In this section we will look at prospect theory and framing effects for values in hopes of understanding why people deviate from the predictions of expected maximization of utility. As a class let’s consider the following example from the Winston Text:

3.1.0.1 Situation 1

$L_1 \text{ 1  $30}$

and

$L_2 \begin{cases} 0.8 \text{  $45} \\ 0.2 \text{  $0} \end{cases}$

Supposedly most people prefer $L_1$ here to $L_2$.

3.1.0.2 Situation 2

Let’s also consider the following two lotteries:

$L_3 \begin{cases} 0.2 \text{  $45} \\ 0.8 \text{  $0} \end{cases}$

and

$L_4 \begin{cases} 0.25 \text{  $30} \\ 0.75 \text{  $0} \end{cases}$

Here supposedly most people will take $L_3$ over $L_4$.

Let $u(0) = 0$ and $u(45) = 1$
From situation 1 can be seen that a decision maker prefers $L_1$ if and only if

$$u(30) > 0.8$$

In situation 2 a decision maker prefers $L_3$ over $L_4$

$$E(U \text{ for } L_3) > E(U \text{ for } L_4)$$

$$\implies 0.8u(0) + 0.2u(45) > 0.25u(30) + 0.75u(0)$$

$$\implies 0.2u(45) > 0.25u(30)$$

$$\implies 0.8 > u(30)$$

Thus, we have a paradox as a person who believes in expected maximization of utility can’t choose both $L_1$ over $L_2$ and $L_3$ over $L_4$.

### 3.1.0.3 Prospect Theory

To explain these results Tversky and Kahneman developed **prospect theory** in the early 1980’s. Here a decision maker has a distorted view of probabilities. Thus, the probability of an event is treated as a distorted probability.

A common way to distort probabilities is to stick them in function such as

$$\Pi(p) = 1.89799p - 3.55995p^2 + 2.662549p^3$$
Using the distorted probability in place of the normal probability when calculating the expected utility of each lottery allows us to find the “prospect” of each lottery. Thus,

- Prospect of $L_1$ : $u(30)$
- Prospect of $L_2$ : $\Pi(0.8)u(45) + 0.2u(0) = 0.60325$
- Prospect of $L_3$ : $\Pi(0.2)u(45) + 0.8u(0) = 0.25850$
- Prospect of $L_4$ : $\Pi(0.25)u(30) + 0.75u(0) = 0.29360u(30)$

Here $L_1$ is preferred to $L_2$ if 

$$u(30) > 0.60325.$$ 

and $L_3$ is preferred to $L_4$ provided:

$$0.2585 > 0.29360u(30) \implies u(30) < 0.880449.$$ 

and we no longer have a paradox!
Example:

Consider the following problem from the Winston Text: A decision maker has a utility function of \( u(x) = x^{1/3} \). A fair coin is flipped and $10 is received for heads, and $0 for tails.

- What is the certainty equivalent for this lottery?
- Using the distorted probability function given in the notes, what is the certainty equivalent for the given lottery.
- Explain why there is a difference in the two certainty equivalents.
- What implication does this have on the method we had discussed in class for estimating a person’s utility function.

Initially we aim to find the certainty equivalent for the lottery. This is a payoff value that makes us indifferent between the payoff and playing the lottery.

\[
E(U \text{ for } L) = \frac{1}{2} u(10) + \frac{1}{2} u(0)
\]

\[
= \frac{1}{2} \left( 10^{1/3} + 0^{1/3} \right)
\]

\[
= \frac{10^{1/3}}{2}
\]

We use this to find a value \( x_{ce} \) such that

\[
(1)u(x_{ce}) = \frac{10^{1/3}}{2} \implies x_{ce} = \left( \frac{10^{1/3}}{2} \right)^3 = \frac{10}{8} = 1.25
\]

and we find the certainty equivalent to the given lottery to be $1.25.

Using the distorted probabilities changes values slightly. Thus,

\[
\text{Prospect for } L = \Pi \left( \frac{1}{2} \right) u(10) + \Pi \left( \frac{1}{2} \right) u(0)
\]

\[
\approx 0.3918261 \left( 10^{1/3} + 0^{1/3} \right)
\]

\[
\approx 0.84416
\]

and a payoff that has the same prospect is given by

\[
\Pi(1)u(x_{ce2}) = u(x_{ce2}) \implies x_{ce2} \approx (0.84416)^3
\]

\[
\implies x_{ce2} \approx 0.60
\]

It makes since that it was lower as according to our distorted view of probability we see events that have a probability of occurring of 1/2 as having a probability that is actually smaller than 1/2, around 0.39.

We should probably estimate the distortion function \( \Pi(p) \) for a decision maker at same time the utility function is estimated.
Chapter 4

Decision Trees

“...well some go this way, and some go that way. But as for me, myself, personally, I prefer the shortcut.”

— Cheshire Cat from the 1951 Disney Alice in Wonderland Film

It is often useful to break a series of large decisions down into a decision tree. This allows the decision maker to view a large complex problem as a series of smaller problems.

4.0.1 Some Basic Notation

When manufacturing a decision tree we will use the following notation:

- **Decision Forks** are used to represent when a decision has a verity of possible outcomes.

  ![Decision Fork Diagram]

- **Event Forks** will be used when outside forces will determine which of several possible outcomes will occur.

  ![Event Fork Diagram]
• A branch of the decision tree is considered a **terminal branch** if no forks emanate from the given branch.

Using this notation we can now set up a decision tree that describes different situations. Consider several examples from the problems section in the Winston Text.

**Example:**

Erica is going to fly to London on August 5 and return home on August 20. It is now July 1, she may buy a one-way ticket (for $350) or a round-trip ticket (for $660). She may also wait until August 1 to buy a ticket. On August 1, a one-way ticket will cost $370, and a round-trip ticket will cost $730. It is possible that between July 1 and August 1, her sister (who works for an airline) will be able to obtain a free one-way ticket for Erica. The probability that her sister will obtain the free one-way ticket is 0.30. If Erica has bought a round trip ticket on July 1 and her sister has obtained a free ticket, she may return “half” of her round trip ticket to the airline. In this case, her total cost would be $330 plus a $50 penalty. Use a decision tree approach to determine how to minimize Erica’s expected cost of obtaining round trip transportation to London.
Example:

Oilco must determine whether or not to drill for oil in the South China Sea. It costs $100,000, and if oil is found the value is estimated to be $600,000. At present, Oilco believes there is a 45% chance that the field contains oil. Before drilling, Oilco can hire (for $10,000) a geologist to obtain more information about the likelihood that the field will contain oil. There is a 50% chance that the geologist will issue a favorable report and a 50% chance of an unfavorable report. Given a favorable report, there is an 80% chance that the field contains oil. Given an unfavorable report, there is a 10% chance that the field contains oil. Determine Oilco’s optimal course of action.
4.0.1.1 Expected Value of Sample Information

As a follow up how can we find the expected value of the sample information (EVSI). In the case of the Oil Company drilling this would be the value we place on the geologists study.

Here we look at the expected value with original information (EVWOI). This can be read from the decision tree by ignoring the branch of the tree where the geologist was hired.

\[
EVWOI = $170,000
\]

We then find the value of the branch where we hire the geologist under the assumption that the geologist’s study didn’t cost anything. This yields the expected value with the sample information.

\[
EVWSI = $190,000
\]

We can then use the difference between the two to tell us the expected value of the sample information.

\[
EVSI = EVWSI - EVWOI
EVSI = $190,000 - $170,000 = $20,000
\]

Thus, the sample information has an expected value of $20,000. This is more than the cost of hiring the geologist, so we should hire the geologist to do the study.
4.0.1.2 Expected Value of Perfect Information

**Perfect Information** means all events still occur with their given probabilities; however we have knowledge of their outcome prior to the decision being made. Thus to find the expected value of perfect information (EVPI) we can calculate the expected value of with perfect information and subtract the expected value with out information.

\[
EVPI = EVWPI - EVWOI
\]

To find the value of \( EVWPI \) we construct a decision tree where all uncertain events occur prior to making our decision.

For the Oil company we then have two events that will occur before the decision is made:

- The geologist’s study (favorable or not favorable).
- Finding oil or not finding oil.

Then we make our decision. This leads to the decision tree:
Decision Trees and Bayes’ Rule

Decision Tree problems lend nicely to a discussion of Bayes’ Rule. Recall from our previous review that when there are \( n \) mutually exclusive world states that make up the set of all possibilities the outcome of an experiment is observed. Giving each world state the label \( S_i \) and assuming that a decision maker has assigned to each state a probability \( P(S_i) \) the outcome of the experiment \( O_j \), is in some way based on the state of the world. Thus,

\[
P(O_j|S_i) = \text{is the likelihood of outcome } O_j \text{ given } S_i
\]

Knowing the likelihood of \( O_j \) given world state \( S_i \) we can now derive an expression for the probability of world state \( S_i \) given we observe outcome \( O_j \) from the experiment. Here we use Bayes’ rule to find posterior probabilities:

\[
P(S_i|O_j) = \frac{P(S_i \cap O_j)}{P(O_j)} = \frac{P(O_j|S_i)P(S_i)}{\sum_{i=1}^{n} P(O_j|S_i)P(S_i)}
\]

Example

Consider Farmer Jones must determine whether to plan corn or wheat. If he plants corn and the weather is warm, he earns $8000; if he plants corn and the weather is cold, he earns $5000. If he plants wheat and the weather is warm, he earns $7000; if he plants wheat and the weather is cold he earns $6500. In the past, 40% of all years have been cold and 60% have been warm. Before planting, Jones can pay $600 for an expert weather forecast. If the year is actually cold, there is a 90% chance that the forecaster will predict a cold year. If the year is actually warm, there is an 80% chance that the forecaster will predict a warm year. How can Jones maximize his expected profits? Additionally find the expected value of the forecaster’s information. What is the expected value of perfect information?
Example

The IUP television network earns an average of $400,000 from a hit show and loses an average of $100,000 on a flop. Of all shows reviewed by the network, 25% turn out to be hits and 75% turn out to be flops. For $40,000, a market research firm will have an audience view a pilot of a prospective show and give its view about whether the show will be a hit or a flop. If a show is actually going to be a hit, there is a 90% chance that the market research firm will predict the show to be a hit. If the show is actually going to be a flop, there is an 80% chance that the market research firm will predict the show to be a flop. Determine how the network can maximize its expected profits. Find the expected value of the research study, and determine the expected value of perfect information.
Chapter 5

Multiple Objectives

"I know the answer! The answer lies within the heart of all mankind! The answer is twelve? I think I'm in the wrong building."

— Peanuts Lucy Van Pelt

Many situations give rise to making a decision where multiple attributes need to be taken into account.

Example:

When purchasing a new vehicle:

- Price
- Gas Mileage
- Seating
- Color
- Number of Doors

Example:

When deciding on a graduate school

- Price
- Name Recognition
- Faculty at School
- Distance to Home
- Size of Program
- Stipend/Tuition
In these cases we need to assign values to each of the attributes, and then sum the value of the attributes in a value or cost function.

- We can typically let $x_i^a = x_i(a)$ be the value of alternative $a$ for attribute $x_i$.
- Then maximizing the value of all the alternatives may be thought of as finding the max of the value function (or similarly minimizing cost):

$$\max_{a \in A} v(x_1(a), x_2(a), \ldots, x_n(a))$$

where $A$ is the set of all possible alternatives.

Some definitions that will ease our discussion of making decisions with multiple objectives include:

- **Additive value function:**

  If there exists $n$ attribute value functions $v_i(x_i)$ such that:

  $$v(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} v_i(x_i)$$

  then $v(x_1, x_2, \ldots, x_n)$ is an additive value function. (Similarly for additive cost functions).

- **Preferentially Independent** or (pi):

  Given two attributes (call them $a_1$ and $a_2$). Here $a_1$ is preferentially independent to $a_2$ provided preferences for values of $a_1$ do not depend on the values of $a_2$.

- **Mutually Preferentially Independent**

  If $a_1$ is pi to $a_2$ and $a_2$ is pi to $a_1$ then $a_1$ is mutually preferentially independent of $a_2$. Or we can write $a_1 \text{ mpi } a_2$.

- A set of attributes $S$ is **mutually preferentially independent** (mpi) of a set of attributes $S'$ if:

  - The values of attributes in $S'$ do not affect preferences for the values of items in $S$.
  - The values of attributes in $S$ do not affect preferences of the values of attributes in $S'$. 

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Theorem 5.0.1 If the set of attributes is mutually preferentially independent, the decision makers preferences can be represented by an additive value function.

See the Winston Text for a reference to the proof, and note that this result is in no manner obvious.

Example:

Consider the following value function with two attributes:

\[ v(x_1, x_2) = x_1 + x_1x_2 + x_2 \]

Note that this is a nonlinear function in \( x_1 \) and \( x_2 \).

If we consider \( x_1 = 6 \), and \( x_2 = 6 \), and compare with a situation where \( x_1 = 4 \) and \( x_2 = 8 \) we obtain:

\[ v(6, 6) = 6 + 36 + 6 = 48 \]
\[ v(4, 8) = 4 + 32 + 8 = 44 \]

Thus, a decision maker will prefer the (6,6) situation to the (4,8) situation.

How do we show that \( x_1 \) is preferentially independent of \( x_2 \)?

Assume that \( a, b, c \geq 0 \) with \( a < b \). If \( a < b \) then we should be able to show that \( v(a, c) < v(b, c) \).

\[ a < b \implies a(1 + c) < b(1 + c) \]
\[ \implies a + ac < b + bc \]
\[ \implies a + ac + c < b + bc + c \]
\[ \implies v(a, c) < v(b, c) \]

which shows that \( x_1 \) is preferentially independent of \( x_2 \). Similarly we could show that \( x_2 \) is preferentially independent of \( x_1 \). This gives \( x_1 \) is mutually preferentially independent from \( x_2 \).

If \( x_1 \) and \( x_2 \) are mutually preferentially independent we should be able to show that \( v(x_1, x_2) \) is an additive value function.
So the question becomes can we represent
\[ v(x_1, x_2) = x_1 + x_1x_2 + x_2 \]
as an additive value function of the form:
\[ v'(x'_1, x'_2) = \sum_{i=1}^{n} v_i'(x'_i) \]
that yields the same preferences for the decision maker? Here we can define some new attributes:
\[ x'_1 = x_1 + x_2 \quad \text{and} \quad x'_2 = x_1 - x_2 \]
with these substitutions it can be seen that:
\[ x_1 = \frac{x'_1 + x'_2}{2} \]
\[ x_2 = x'_1 - \left( \frac{x'_1 + x'_2}{2} \right) \]
Using these substitutions in the original value function yields:
\[ v'(x'_1, x'_2) = x'_1 + \frac{x'_1^2}{4} - \frac{x'_2^2}{4} \]
an additive value function that gives the same value to a decision makers preferences.
The idea now becomes to extend our concept of utility functions to multiple objective decisions. In doing so we can include a decision makers attitude toward risk. Here we will use:

- A **multiattribute utility function** where one or more attribute affects a decision makers preference.

By defining $x_i$ to be the level of attribute $i$,

$$ u(x_1, x_2) $$

is our utility function with attribute 1 at level $x_1$ and attribute 2 at level $x_2$.

Finding a utility function for $u(x_1, x_2, \ldots, x_n)$ in general is a very difficult task. For this class we will make some assumptions:

- Attribute 1 is **utility independent** of attribute if preferences for lotteries involving different levels of attribute 1 do not depend on the level of attribute 2.

Consider the lotteries where we have access to a supercomputer for a specified number of hours, and a cash payout of a specified amount.

And suppose we are indifferent between the following lotteries:

$$ L_1 \quad 1 \quad 10\text{hrs} \; , \; 500 $$

and

$$ L_2 \quad 0.5 \quad 15\text{hrs} \; , \; 500 $$

$$ \quad 0.5 \quad 5\text{hrs} \; , \; 500 $$

If the time on the super computer is truly utility independent of the cash pay out we should also be indifferent between:

$$ L_3 \quad 1 \quad 10\text{hrs} \; , \; 1000 $$

and

$$ L_4 \quad 0.5 \quad 15\text{hrs} \; , \; 1000 $$

$$ \quad 0.5 \quad 5\text{hrs} \; , \; 1000 $$
So a certainty equivalent for a lottery where we have a 50/50 chance of 5 hours or 15 hours on a super computer would be a guaranteed 10 hours.

We have also illustrated that hours on the super computer is utility independent to the cash payout.

- If attribute 1 is ui of attribute 2, and attribute 2 is ui of attribute 1 then we say the two attributes are mutually utility independent (mui).

We can also state that a decision makers utility function is a multilinear function of the form:

\[ u(x_1, x_2) = k_1u_1(x_1) + k_2u_2(x_2) + k_3u_1(x_1)u_2(x_2) \]

if and only if attributes 1 and 2 are mutually utility independent.

We can illustrate utility independence using the lotteries defined above. Assuming that we are indifferent between lotteries \( L_1 \) and \( L_2 \) then the expected utility of these lotteries is the same. Doing some algebra and assuming the decision maker has a multilinear utility function allows for us to show the decision maker will be indeed indifferent between lotteries \( L_3 \) and \( L_4 \).

The decision makers utility function can be further simplified if it is has additive independence between attributes. Assuming the best possible outcome of each attribute is denoted by “best”, and the least favorable value of an attribute is denoted by “worst”. A decision maker will have a utility function that has additive independence provided the decision maker is indifferent between the following lotteries where we have an ordered pair of rewards for attributes \((x_1, x_2)\):

\[
\begin{align*}
L_1 & \quad 0.5 \quad \text{(worst, worst)} \\
& \quad 0.5 \quad \text{(best, best)}
\end{align*}
\]

and

\[
\begin{align*}
L_2 & \quad 0.5 \quad \text{(worst, best)} \\
& \quad 0.5 \quad \text{(best, worst)}
\end{align*}
\]

This will allow us to see that the value of \( k_3 \) in the decision maker’s utility function is zero. To illustrate this lets assign values to the utility of the best and worst rewards for each attribute of a possible decision. Then,

\[ u_1(\text{best}) = 1, u_1(\text{worst}) = 0 \]
\[ u_2(\text{best}) = 1, u_2(\text{worst}) = 0 \]

Using the defined multilinear utility function (where attributes \( x_1 \) and \( x_2 \) are assumed to be mutually utility independent) yields the following utility of the different outcomes for each branch of the above defined Lotteries:

\[
\begin{align*}
    u(\text{best, best}) &= k_1 + k_2 + k_3 \\
    u(\text{worst, worst}) &= 0 \\
    u(\text{best, worst}) &= k_1 \\
    u(\text{worst, best}) &= k_2
\end{align*}
\]

Assuming indifference between lotteries \( L_1 \) and \( L_2 \) and considering the expected utility of the lotteries allows us to write:

\[
\begin{align*}
    0.5u(\text{best, best}) + 0.5u(\text{worst, worst}) &= 0.5u(\text{best, worst} + 0.5u(\text{worst, best}) \\
    \Rightarrow 0.5(k_1 + k_2 + k_3) + 0.5(0) &= 0.5k_1 + 0.5k_2 \\
    \Rightarrow k_3 &= 0
\end{align*}
\]

Then we can say that for a decision maker who is mutually utility independent between two attributes, that also exhibits additive independence has a utility function of the form:

\[
u(x_1, x_2) = k_1 u_1(x_1) + k_2 u_2(x_2).\]

### 5.0.2 Calculating Utility Functions

If a decision maker is mutually utility independent between two attributes, we can determine the values of \( u_1(x_1) \), \( u_2(x_2) \), \( k_1 \), \( k_2 \), and \( k_3 \) in the same manner use for finding a utility function with only one attribute. Fixing the reward value for one of the attributes say \( x_2 \). Then we can find the value for \( x_{1/2} \) by setting its value such that the decision maker is indifferent between:

\[
L \quad \quad 1 \quad \quad x_{1/2}, x_2
\]

and

\[
L' \quad 0.5 \quad \text{worst, } x_2 \\
L' \quad 0.5 \quad \text{best, } x_2
\]
Note this gives us the certainty equivalent for the reward value of attribute $x_1$ for the lottery $L'$. To find the values of the constants $k_1, k_2$, and $k_3$ we can use a rescaling of the utility functions $u_1(x_1), u_2(x_2)$ and $u(x_1, x_2)$ such that

\[ u_1(\text{best}) = 1, u_1(\text{worst}) = 0, u_2(\text{best}) = 1, u_2(\text{worst}) = 0, \]
\[ u(\text{best}, \text{best}) = 1 \text{ and } u(\text{worst}, \text{worst}) = 0 \]

Then from our multilinear utility function we can see that:

\[
\begin{align*}
  u(\text{best}, \text{worst}) &= k_1 u_1(\text{best}) + k_2 u_2(\text{worst}) + k_3 u_1(\text{best}) u_2(\text{worst}) \\
  &= k_1
\end{align*}
\]

Then we can use the fact that a mutually utility indifferent decision maker will be indifferent between the following lotteries to find the value of $k_1$.

\[
\begin{align*}
  1 & u(\text{best}, \text{worst}) = k_1
\end{align*}
\]

and

\[
\begin{align*}
  (1 - k_1) & u(\text{worst}, \text{worst}) \\
  k_1 & u(\text{best}, \text{best})
\end{align*}
\]

Similarly the value of $k_2$ is found such that the decision maker is indifferent between the lotteries:

\[
\begin{align*}
  1 & u(\text{worst}, \text{best})
\end{align*}
\]

and

\[
\begin{align*}
  (1 - k_2) & u(\text{worst}, \text{worst}) \\
  k_2 & u(\text{best}, \text{best})
\end{align*}
\]

To find $k_3$ note that the multilinear utility function and the rescaling of

\[
\begin{align*}
  u_1(\text{best}) = 1, u_2(\text{best}) = 1, \text{ and } u(\text{best}, \text{best}) = 1
\end{align*}
\]

\[
\begin{align*}
  & \implies k_1 + k_2 + k_3 = 1 \\
  & \implies k_3 = 1 - k_1 - k_2.
\end{align*}
\]
Summary

We can come up with a reasonable multi-attribute utility function provided we have mutually utility independent attributes.

• Check to see if the utility function has additive independence for further simplification.
• Use Lottery method to find the values for $u_1(x_1)$ and $u_2(x_2)$.
• Determine values for $k_1$, $k_2$, (and $k_3$ if there is no additive independence).
• You can check the utility function created by seeing if it reflects the decision makers preferences for a series of lotteries. The Winston text has a nice illustrating example of this.

The following series of example problems are all from the Winston Text.

Example

National Express Carriers is interested in two attributes:

• The average cost of delivering a letter (know to be between $1 and $5).
• Percentage of all letters reaching their destination on time (known to be between 70% and 100%).

1. Would National’s multiattribute utility function exhibit mutual utility independence (mui)?

2. Would National’s utility function be additive?

3. Assuming that the two attributes listed are mutually utility independent. Suppose that National is indifferent between ($1, 70\%$) for certain and the following situation:

   \[
   \begin{array}{c c c c c c}
   0.3 & \rightarrow & (1, 100\%) \\
   \downarrow & & \downarrow \\
   0.70 & \rightarrow & (5, 70\%)
   \end{array}
   \]

   Also assume that National is indifferent between ($5, 100\%$) for certain and a situation where

   \[
   \begin{array}{c c c c c c}
   0.5 & \rightarrow & (1, 100\%) \\
   \downarrow & & \downarrow \\
   0.5 & \rightarrow & (5, 70\%)
   \end{array}
   \]
Find National’s multi-attribute utility function.

For the first two questions posed we would really need to know more information.

Note from the Lotteries given that the value for $k_1 = 0.3$ and the value for $k_2 = 0.5$. Using the fact that $k_3 = 1 - k_1 - k_2$ yields that $k_3 = 0.2$. This gives the utility function as:

$$u(x_1, x_2) = 0.3u(x_1) + 0.5u_2(x_2) + 0.2u_1(x_1)u_2(x_2)$$

Note we do not have a definition for $u_1(x_1)$ or $u_2(x_2)$. 
Example:

Gotham City is trying to determine how many ambulances it should have and how to staff them. Each ambulance may be staffed with paramedics or emergency medical technicians. Paramedics are considered to provide better service and are paid higher salaries. Budgetary limitations have forced the city to choose between the following two alternatives:

- Four ambulances, two staffed with emergency medical technicians and two staffed with paramedics.
- Three ambulances, all staffed with paramedics.

The city authorities believe that the following two attributes determine the city’s satisfaction with ambulance service:

- Time until an ambulance reaches a patient.
- Percentage of ambulances calls handled by paramedics.

Assume that the decision makers of Gotham City exhibit mutually utility independence between these attributes and that the utility function for each is:

\[ u_1(x_1) = 1 - \frac{x_1^2}{900} \quad \text{and} \quad u_2(x_2) = \frac{x_2^2}{10,000} \]

The time for an ambulance to reach a patient is always between 0 and 30 minutes. The city authorities are indifferent between a certain 30 minute arrival time paired with 100% of calls being handled by paramedics, and a situation in which

\[ 0.40 \quad (0, 100\%) \]
\[ 0.60 \quad (30, 0\%) \]

Additionally the city authorities are also indifferent between a situation where with certainty it takes no time to reach the patient coupled with no patients being treated by paramedics and a situation in which

\[ 0.80 \quad (0, 100\%) \]
\[ 0.20 \quad (30, 0\%) \]
Assume that if there is an ambulance available when a call comes in the ambulance will arrive within 5 minutes; if an ambulance is not available when a call comes in, it will arrive in 20 minutes.

With three ambulances one will be immediately available 60% of the time. With four ambulances one will be immediately available 80% of the time.

- Determine the city authorities’ multiattribute utility function.
- Which alternative should they choose?
Example:

Let $x_1$ and $x_2$ be the undergraduate grade point average (GPA) of a student applying to a state university’s MBA program, and GMAT score of the same student respectively. Suppose that the preference between applicants is based on the following value function:

$$v(x_1, x_2) = 200x_1 + x_2 - 0.1x_2x_1^2$$

- Would the MBA program prefer a student with a 3.8 GPA and a 500 GMAT score to a student with a 3.0 GPA and a 710 GMAT score?

- Does the value function exhibit mutual preferential independence?

Answering the first question is a simple case of plugging into the given value function.

The second question is a bit more interesting. In order to show that the value function exhibits mutual preferential independence we can consider for all fixed values of $x_2$ with $a < b$ we should be able to show that $v(a, x_2) < v(b, x_2)$. Similarly for any fixed value of $x_1$ with $c < d$ we should be able to show $v(x_1, c) < v(x_1, d)$. If both conditions can be seen we have demonstrated that the value function exhibits mutual preferential independence.

Let’s assume that $a < b$ (say $a = 1$ and $b = 2$). Now consider the given value function and note

$$200 + x_2 - 0.1x_2 < 200(2) + x_2 - 0.1x_2(4)$$

$$\implies 200 - 0.1x_2 < 400 - 0.4x_2$$

$$\implies 0.3x_2 < 200$$

$$\implies x_2 < \frac{200}{0.3}$$

and the inequality will fail as soon as $x_2$ is chosen large enough. This gives a contradiction to the given value function being mutually preferentially independent.
Example:

At present, littering is punished by a $50 fine, and there is a 10% chance that a litterer will be brought to justice. To cut down on littering, Gotham City is considering two alternatives:

- Raise the littering fine by 20% (to $60).
- Hire more police and increase by 20% the probability that a litterer will be brought to justice (to a 12% probability).

Assuming that the people of Gotham are risk-averse, which alternative will lead to a larger reduction in littering?

In this instance we are looking at comparing two lotteries:

Raise Fine:

\[
L_{rf} = \begin{cases} 
0.1 & -60 \text{ caught} \\
0.9 & 0 \text{ not caught}
\end{cases}
\]

Hire More Police:

\[
L_{hp} = \begin{cases} 
0.12 & -50 \text{ caught} \\
0.88 & 0 \text{ not caught}
\end{cases}
\]

If the people of Gotham where risk neutral they would have a utility value for reward \( x \) given by:

\[
u(x) = \left( \frac{u(0) - u(-60)}{60} \right) x + u(0)
\]

\[
\implies u(-50) > \frac{5}{6} u(0) + \frac{5}{6} u(-60) + u(0)
\]

\[
\implies u(-50) > \frac{5}{6} u(-60) + \frac{1}{6} u(0)
\]

Multiply the end result by 0.12 and add 0.88\( u(0) \) to both sides giving:

\[
0.12u(-50) + 0.88u(0) > 0.1u(-60) + (0.02 + 0.88)u(0)
\]

\[
\implies 0.12u(-50) + 0.88u(0) > 0.1u(-60) + 0.9u(0)
\]

\[
\implies E(U \text{ for } L_{hp}) > E(U \text{ for } L_{rf})
\]

This show that the City should raise the litter fine as it has a lower expected utility for a risk-averse Gotham citizens, and would be a stronger deterrent from their nasty littering ways.
OUTLINE OF TEST TOPICS

- Review of Basic Calculus

- Basic Probability
  - mutually exclusive
  - conditional probability
    \[ P(E_2|E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} \]
  - Bay’s Rule

- Random Variables
  - Mean
  - Variance
  - Distribution Functions
  - Normal Distributions

- Decision Making Under Uncertainty
  - Dominated Actions
  - Different Decision Criterion (4 different types)
  - Utility Theory
  - Lotteries, and Compound Lotteries
  - Certainty equivalents
  - Risk Premium: (expected value - certainty equivalent)
  - Attitude toward risk: How can it be seen in utility function?
  - Prospect Theory
  - Decision Trees (expected value, utility of, prospect of).
  - Bay’s Rule and Decision Trees: prior probabilities, likelihoods.
  - Multiple Attribute Decision Making.
Chapter 6

Markov Chains

"The world always seems brighter when you’ve just made something that wasn’t there before."

—Neil Gaiman

When studying the topic of Markov Chains it may be useful to do some simulations. Recall that we all have access to the IUP sage server at:

http://sage.nsm.iup.edu:8000

For these notes we will follow the notes presented in Chapter 17 of the Winston Text.

6.1 Stochastic Process

Consider looking at a systems characteristics at discrete times:

\[ t = 0, 1, 2, \ldots \]

and we let \( X_t \) be the value of a specific system characteristic at time \( t \). The observation, \( X_t \), at time \( t \) is generally not know before time \( t \).

A discrete-time stochastic process is a description of the relationship between the random variables \( X_i \) for \( i \in \{0, 1, 2, \ldots\} \).
Example:

The Gambler's Ruin Consider playing a game where with odds $p$ you will win a dollar, and with odd $(1 - p)$ you will lose a dollar. The game is played until you double your winnings, or you have no money left to play. We can let $X_i$ represent our capital position at time $i$.

If for instance we start playing the game with three dollars $X_0 = 3$, we will quite when we have nothing or accumulated $6$. Here the string:

$$X_0, X_1, X_2, \ldots, X_t$$

may be viewed as a discrete time stochastic process. Note

- with probability $p$ we will have $4$ after one play of the game: $(X_1 = 4)$.
- with probability $p - 1$ we will have $2$ after one play of the game: $(X_1 = 2)$.
- If $X_t = 0$ then $X_{t+1}$ and all later times will also have value 0.
- If $X_t = 6$ then $X_{t+1}$ and all later times will have value 6.

The following is a simple sage code to simulate the Gamblers Ruin Stochastic Process.

```
StartValue = 2
HouseOdds = 0.5
QuitWins = 8
P = [HouseOdds, (1.0-HouseOdds)]
X = GeneralDiscreteDistribution(P)
Terminate = 1
PlayCounter = 0
Value = StartValue

print Value
# Start Stochastic Process.
while(Terminate != 0):
    PlayCounter = PlayCounter + 1;
    x = X.random_element() 
    if(x == 0):
        Value = Value + 1;
    if(x == 1):
        Value = Value - 1;
    print Value;
    if ((Value == 0) or (Value == QuitWins)):
        Terminate = 0;
print "Value = ", Value
print "After ", PlayCounter , " Plays of the game."
```
Example:

Stock Price If we let $X_0$ be the value of a share of stock (say in Jamestown Ice and Storage) at the start of the current trading day. We could also let $X_t$ be the value of the share on the beginning of the $t^{th}$ trading day in the future. If we know the value of $X_0, X_1, X_2, \ldots, X_t$ we will know something about the distribution of $X_{t+1}$.

If we can observe the value of a stochastic process at any time (not just predetermined instants) we will consider this a continuous-time stochastic process. An example here could be the number of people in Stright Hall at any instant. If we view the value of a stock price as a continuous time stochastic process we may obtain the famous Black-Sholes option pricing formula [7]

6.1.1 Markov Chains

A special type of discrete-time stochastic process is called Markov Chain. Assume that at any time $t$ our stochastic process can be in any one of $s$ distinct discrete states labeled:

$$1, 2, \ldots, s$$

Here a discrete-time stochastic process is a Markov Chain if for $t = 0, 1, 2, \ldots$ and all states:

$$P(X_{t+1} = i_{t+1}|X_t = i_t, X_{t-1} = i_{t-1}, \ldots, X_1 = i_1, X_0 = i_0) = P(X_{t+1} = i_{t+1}|X_t = i_t).$$

and hence only depends on the previous state, and not the entire chain of events that it passed through to get to that state.

We will also make the assumption that for all states $i$ and $j$ and all times $t$ that

$$P(X_{t+1} = j|X_t = i)$$

is independent of $t$.

This allows us to denote:

$$p_{ij} = P(X_{t+1} = j|X_t = i) \quad (6.1.1)$$

the probability that the system is in state $j$ at step $t + 1$ given it was in state $i$ at step $t$.

- Moving from state $i$ to state $j$ is called a transition, and the values of $p_{ij}$ are known as the transition probabilities of the Markov Chain.

- Note that (6.1.1) implies the probabilities of transitioning from state $i$ to state $j$ do not vary with time. Thus, (??) is sometimes called the Stationary Assumption.
The initial probability distribution for a Markov Chain is a vector \( q \) where \( q_i \) denotes the probability that the system starts in state \( i \).

\[ q_i = P(X_0 = i) \]

In most applications the transition probabilities are displayed in an \( s \times s \) matrix \( P \).

\[
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1s} \\
p_{21} & p_{22} & \cdots & p_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
p_{s1} & p_{s2} & \cdots & p_{ss}
\end{bmatrix}
\]

Given that the system is in state \( i \) at time \( t \) and that it must also be somewhere at time \( t + 1 \) we note that:

\[
\sum_{j=1}^{s} P(X_{t+1} = j|X_t = i) = 1
\]

or equivalently

\[
\sum_{j=1}^{s} p_{ij} = 1.
\]

Note the following:

- The values of \( P \) are all non-negative.
- The sum of the entries in each row of \( P \) is 1.

Example:

What would the transition matrix for the Gamblers Ruin as stated above \((X_0 = 3)\) look like?

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-p+1 & 0 & p & 0 & 0 & 0 & 0 \\
0 & -p+1 & 0 & p & 0 & 0 & 0 \\
0 & 0 & -p+1 & 0 & p & 0 & 0 \\
0 & 0 & 0 & -p+1 & 0 & p & 0 \\
0 & 0 & 0 & 0 & -p+1 & 0 & p \\
0 & 0 & 0 & 0 & 0 & -p+1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Note here that each row of the represents a current state, and each column in a given row represents the likely hood of entering the column indexed state at the next step of the chain. What would a graphical representation of the transition matrix look like?
Example:

In Smalltown 90% of all sunny days are followed by sunny days, and 80% of all cloudy days are followed by cloudy days. Use this information to model Smalltown’s weather as a Markov chain. Here letting the state for a day be the day’s weather yields two states with a transition matrix:

\[ S = \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} \]

. Here the transitions are from state \( i \) (row) to state \( j \) (column).

Can we extend this two a multiple day prediction?

Example:

Now assume that tomorrow’s weather depends on the last two days of Smalltowns’ weather in the following manner:

- If the last two days have been sunny, the 95% of the time, tomorrow will be sunny.
- If yesterday was cloudy and today is sunny, then 70% of the time, tomorrow will be sunny.
- If yesterday was sunny and today is cloudy then 60% of the time tomorrow will be cloudy.
- If the last two days have been cloudy, then 80% of the time, tomorrow will be cloudy.

How can we model Smalltown’s weather as a Markov Chain now?

If the weather depended on the last three days how many states do we need in our Markov chain model to forecast the weather?

For the states of the system we now need to consider a two day event. Thus the states letting \( S \) denote sunny, and \( C \) denote cloudy:

\[(\text{yesterday, today}) = \{(S, S), (S, C), (C, S), (C, C)\}\]

The transition matrix then becomes:

\[ P = \begin{pmatrix} 0.95 & 0.05 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.4 & 0.6 \\ 0.7 & 0.3 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.2 & 0.8 \end{pmatrix} \]

Note we are going from the row state to the column state. We can use this technique to model a discrete time stochastic process that depends on multiple prior states as a Markov chain. In the case of three days we would need \( 2^3 \) possible states to model all possible three day weather patterns.
6.1.2 \( n \)-Step Transitions

We can now consider \( n \)-step transition probabilities. That is if a Markov chain is in state \( i \) at the end of the \( m^{th} \) time step, what is the probability that it is in state \( j \), \( n \) time steps later?

\[
P(X_{m+n} = j | X_m = i)
\]

Note this is the same as

\[
P(X_n = j | X_0 = i) = P_{ij}(n)
\]

where \( P_{ij}(n) \) is called the \( n \)-step probability of a transition from state \( i \) to state \( j \).

- We know that \( P_{ij}(1) = p_{ij} \).

- To find \( P_{ij}(2) \) note that we must go from state \( i \) to some other state \( k \) and then from state \( k \) to state \( j \). So logically we have

\[
P_{ij}(2) = \sum_{k=1}^{s} (\text{Prob. of Transition from } i \text{ to } k) (\text{Prob. of Transition from } k \text{ to } j).
\]

\[
= \sum_{k=1}^{s} p_{ik}p_{kj} \tag{6.1.2}
\]

Note that the RHS of (6.1.2) is the dot product of Row \( i \) of \( P \) with column \( j \) of \( P \) (where \( P \) is the transition matrix for our stochastic process).

- We can see then that \( P_{ij}(n) \) is the \( ij^{th} \) element of \( P^n \).

- For \( P_{ij}(0) \) note

\[
P_{ij}(0) = \begin{cases} 
1 & \text{if } j = i \\
0 & \text{if } j \neq i
\end{cases}
\]

as \( P_{ij}(0) = P(X_0 = j | X_0 = i) \).

If we consider again the Gamblers Ruin problem where we fix the odds of winning to be 0.5 starting the game with $3. What is the probability that we end the game with $6 in 3 turns?

\[
P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

66
Considering three turns the transition probabilities are given by:

\[
P^3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Thus, we can see the odds of ending the game in three turns with six dollars has probability \( \frac{1}{8} \).

Note that for large values of \( n \) the probability of being in a given state seems to settle into a value, and is less dependent on the initial state of the Markov Chain. This will be explored further in the forthcoming notes.

### 6.1.3 Classification of States

In order to discuss Markov chains further we need to define some terms. Consider a stochastic process with the following graph:

This stochastic process has a transition matrix given by:

\[
P = \begin{pmatrix}
\frac{3}{10} & \frac{2}{5} & 0 & 0 & 0 \\
\frac{1}{10} & 0 & 0 & \frac{3}{10} & 0 \\
0 & 0 & \frac{7}{10} & 0 & \frac{3}{10} \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
\end{pmatrix}
\]

We will use this stochastic process and associated transition matrix to illustrate some of the following definitions.
• **path**: Given two states $i$ and $j$ a path is a sequence of transitions such that it begins in $i$ and ends in $j$ where each transition in the sequence has a positive probability of occurring.

• **reachable**: A state $j$ is considered reachable from state $i$ if there is a path from state $i$ to state $j$.

• **communicate**: Two states $i$ and $j$ are said to communicate if $i$ is reachable from $j$ and $j$ is reachable from $i$.

  Note for our example that states 1, 2, and 4 all communicate. Also 3 and 5 communicate; however, 1 does not communicate with 3, and so on.

• **closed set**: A set of states is closed if no state outside the set is reachable from a state inside the set.

  Note that the states 1, 2, and 4 are a closed set.

• **absorbing state**: A state is said to be absorbing if $p_{ii} = 1$.

  In our Gambler’s Ruin example the broke state, and the winning state were absorbing. Once the state is entered the system does not leave that state.

• **transient state**: A state $i$ is transient if there exists a state $j$ that is reachable from $i$; however state $i$ is not reachable from $j$.

  Note here we have a way to leave state $i$ that never will return to state $i$. In the Gambler’s Ruin any state that was not “broke” or “win” was a transient state.

• **recurrent**: A state that is not transient is considered to be recurrent.

  Every state in our example stochastic process is recurrent.

• **periodic**: A state $i$ is periodic with period $k$ when $k$ is the smallest number such that all paths that return to state $i$ have a length that is a multiple of $k$.

  A stochastic process with the following transition matrix is periodic:

  $$Q = \begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  1 & 0 & 0 
\end{pmatrix}$$

  Note non-periodic states are considered **aperiodic**.

• **ergodic** A chain where all the states are aperiodic, recurrent, and communicate is considered **ergodic**.
Consider the following problem from the Winston text:

1. Consider the following transition matrix:
\[
P = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3}
\end{pmatrix}
\]

(a) Which states are transient?
State 4 is transient. Note that once you leave you never come back.

(b) Which states are recurrent?
States 1, 2, 3, 5, and 6 are all recurrent states for the given system.

(c) Identify the closed sets of states.
Here 1, 3, and 5 are a closed set. Additionally 2, and 6 are also a closed set.

(d) Is the chain ergodic?
Since state 4 is transient, the system can not be ergodic.

6.1.4 Steady State Probabilities

We can now examine the long run behavior of a Markov Chain allowing for us to describe the steady state probabilities of a Markov Chain.

Theorem 6.1.1 Let \( P \) be a transition matrix for an \( s \)-state ergodic chain. Then there exists a vector
\[
\pi = [\pi_1 \ \pi_2 \ \ldots \ \pi_s]
\]
such that
\[
\lim_{n\to\infty} P^n = \begin{pmatrix}
\pi_1 & \pi_2 & \ldots & \pi_s \\
\pi_1 & \pi_2 & \ldots & \pi_s \\
\vdots & \vdots & \ddots & \vdots \\
\pi_1 & \pi_2 & \ldots & \pi_s
\end{pmatrix}
\]

Thus, for any initial state \( i \) we have
\[
\lim_{n\to\infty} P_{ij}(n) = \pi_j
\]
and regardless of the initial state after a long time the probability that we are in any given state \( j \) of the chain is \( \pi_j \).

The vector \( \pi \) is often called the **steady-state distribution** or the **equilibrium distribution** of the Markov chain.
How do we go about finding the stead-state distribution?

Note that for large values of $n$ and any state $i$ that

$$P_{ij}(n+1) \approx P_{ij}(n) \approx \pi_j$$

(6.1.3)

Note that

$$P_{ij}(n+1) = \text{(row i of } P^n) \cdot \text{(column j of } P)$$

$$= \sum_{k=1}^{s} P_{ik}(n)p_{kj}$$

Note that for a large enough value of $n$ we can substitute (6.1.3) in and achieve:

$$\pi_j = \sum_{k=1}^{s} \pi_k p_{kj}$$

$$\implies \pi = \pi P$$

There is a small flaw in this system! If we proceed with the system as written we will be looking for a zero vector on the right hand side.

This flaw can be fixed by noting that the sum of the steady-state probabilities is 1. Thus, we and replace any of the equation in

$$\pi = \pi P$$

with the equation

$$\pi_1 + \pi_2 + \cdots + \pi_s = 1$$

and we may solve for the steady state probabilities.

**Example:**

Consider the following example form the Winston Text: Two types of squirrels (gray and black) have been seen in Pine Valley. At the beginning of each year, we determine which of the following is true:

- There are only gray squirrels in Pine Valley.
- There are only black squirrels in Pine Valley.
- There are both gray and black squirrels in Pine Valley.
- There are no squirrels in Pine Valley.
Over the course of many years the following transition matrix has been estimated (ordering
the rows and columns as: Gray, Black, Both, Neither).

\[ P = \begin{pmatrix}
0.7 & 0.2 & 0.05 & 0.05 \\
0.2 & 0.6 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.8 & 0.0 \\
0.05 & 0.05 & 0.1 & 0.8 \\
\end{pmatrix} \]

1. During what fraction of years will gray squirrels be living in Pine Valley?

2. During what fraction of years will black squirrels be living in Pine Valley?

Before proceeding to find the steady state distribution lets consider finding the values of \( P^{10} \)
and \( P^{20} \).

\[ P^{10} = \begin{pmatrix}
0.294 & 0.243 & 0.269 & 0.194 \\
0.288 & 0.239 & 0.277 & 0.196 \\
0.292 & 0.244 & 0.308 & 0.156 \\
0.261 & 0.22 & 0.288 & 0.23 \\
\end{pmatrix}, \quad \text{and} \quad P^{20} = \begin{pmatrix}
0.286 & 0.238 & 0.285 & 0.191 \\
0.286 & 0.238 & 0.285 & 0.191 \\
0.287 & 0.239 & 0.286 & 0.188 \\
0.285 & 0.237 & 0.286 & 0.192 \\
\end{pmatrix} \]

To find the steady state distribution exactly we consider:

\[ \pi P = \pi \]

\[ \implies [\pi_1 \pi_2 \pi_3 \pi_4] \begin{pmatrix}
0.7 & 0.2 & 0.05 & 0.05 \\
0.2 & 0.6 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.8 & 0.0 \\
0.05 & 0.05 & 0.1 & 0.8 \\
\end{pmatrix} = [\pi_1 \pi_2 \pi_3 \pi_4] \]

Here we can set up a system of equations:

\[ \begin{pmatrix}
-0.3 & 0.2 & 0.1 & 0.05 \\
0.2 & -0.4 & 0.1 & 0.05 \\
0.05 & 0.1 & -0.2 & 0.1 \\
1.0 & 1.0 & 1.0 & 1.0 \\
\end{pmatrix} \begin{pmatrix}
\pi_1 \\
\pi_2 \\
\pi_3 \\
\pi_4 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\end{pmatrix} \]

We augment the system using

\[ \sum_{k=1}^{s} \pi_k = 1 \]

in place of any equation of our liking. This yields the system:

\[ \begin{pmatrix}
-0.3 & 0.2 & 0.1 & 0.05 \\
0.2 & -0.4 & 0.1 & 0.05 \\
0.05 & 0.1 & -0.2 & 0.1 \\
1.0 & 1.0 & 1.0 & 1.0 \\
\end{pmatrix} \begin{pmatrix}
\pi_1 \\
\pi_2 \\
\pi_3 \\
\pi_4 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
\end{pmatrix} \]
Resulting in:

$$\pi = (0.285714285714, 0.238095238095, 0.285714285714, 0.190476190476)$$

So we can answer the two question presented to us: gray squirrels live in Pine Valley 57.1428571429 percent of the time, and black squirrels live in Pine Valley 52.380952381 percent of the time.

**Observations:**

The time before a Markov chain reaches a steady state behavior is called the transient (or short-run) behavior. For the short run behavior we use the given formulas for

$$P_{ij}(n)$$

and once $n$ is large enough we can consider the values of $\pi$ as accurately describing the probability of being in any given state.

**Interpretation**

Note that at the steady-state probabilities may be thought of as follows:

$$\pi = \pi P$$

or term by term as:

$$\pi_j = \sum_{k=1}^{s} \pi_k p_{kj}.$$ 

Subtract from both sides the probability that the chain is in state $j$ and transitions at the next step to state $j$ (that is subtract $\pi_j p_{jj}$).

Thus,

$$\pi_j (1 - p_{jj}) = \sum_{k \neq j} \pi_k p_{kj}$$

What does each side represent?

- The left hand side denotes the probability of leaving state $j$.
- The right hand side denotes the probability of entering state $j$.
- The steady-state probability is often called the equilibrium probability.
6.1.5 Mean First Passage Time

It is often useful to get a handle on the expected number of transitions a chain must pass through before it reaches a given state.

Given that a chain is currently in state $i$

$m_{ij}$ will denote the mean first passage time form state $i$ to state $j$.

Make note that

$$p_{ij} + \sum_{k \neq j} p_{ik} = 1$$

Then we may find the expected number of step to transition from state $i$ to state $j$ by looking at:

$$m_{ij} = (1)p_{ij} + \sum_{k \neq j} p_{ik}(1 + m_{kj})$$

$$= p_{ij} + \sum_{k \neq j} p_{ik}(1 + m_{kj})$$

$$= p_{ij} + \sum_{k \neq j} p_{ik} + \sum_{k \neq j} p_{ik}m_{kj}$$

$$= 1 + \sum_{k \neq j} p_{ik}m_{kj}$$

Note we may consider this expression in matrix form by denoting:

$N_j$ = the probability transition matrix without row and column $j$.

Then $m$ denotes an $(s - 1) \times 1$ column vector.

$1$ = is an $(s - 1) \times 1$ column vector of ones.

$m_j$ = the $(s - 1) \times 1$ column vector that yields the mean first passage from any state into state $j$. This gives:

$$m_j = 1 + N_j m$$

$$\Rightarrow 1 + N_j m_j = m_j$$

$$\Rightarrow N_j m_j - m_j = -1$$

$$\Rightarrow (N_j - I)m_j = -1$$

$$\Rightarrow (I - N_j)m_j = 1$$

$$\Rightarrow m_j = (I - N_j)^{-1} 1$$
Consider finding the steady state distribution, as well as the mean first passage times for a Markov chain with the following transition matrix:

\[
P = \begin{pmatrix}
0.3 & 0.6 & 0.1 \\
0.1 & 0.6 & 0.3 \\
0.05 & 0.4 & 0.55
\end{pmatrix}
\]

Let's start by considering the steady state probabilities. Thus, we are looking to find the vector \( \pi \) such that

\[
\pi = \pi P
\]

\[
\Rightarrow \begin{pmatrix}
\pi_1 \\
\pi_2 \\
\pi_3
\end{pmatrix} = \begin{pmatrix}
\pi_1 \\
\pi_2 \\
\pi_3
\end{pmatrix} \begin{pmatrix}
0.3 & 0.6 & 0.1 \\
0.1 & 0.6 & 0.3 \\
0.05 & 0.4 & 0.55
\end{pmatrix}
\]

\[
\Rightarrow (P^T - I)\pi^T = 0
\]

Replacing one row of the system with the constraint that

\[
\sum_{i=1}^{3} \pi_i = 1
\]

\[
\Rightarrow \begin{pmatrix}
-0.7 & 0.1 & 0.05 \\
0.6 & -0.4 & 0.4 \\
1.0 & 1.0 & 1.0
\end{pmatrix} \begin{pmatrix}
\pi_1 \\
\pi_2 \\
\pi_3
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

\[\pi = (0.101694915254, 0.525423728814, 0.372881355932)\]

Next we can consider the mean first passage time. Consider passing from states 2 and 3 into state 1.

\[m_1 = \begin{pmatrix}
m_{21} \\
m_{31}
\end{pmatrix}\]

Here we solve the system

\[(I - N_1)m_1 = 1\]

\[
\Rightarrow \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} - \begin{pmatrix}
0.6 & 0.3 \\
0.4 & 0.55
\end{pmatrix} \begin{pmatrix}
m_{21} \\
m_{31}
\end{pmatrix} = \begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

\[m_1 = \begin{pmatrix}
12.5 \\
13.3333333333
\end{pmatrix}\]

Thus, it will take about 12.5 transitions to get to state 2 from state 1, and approximately 13.33 transitions to move from state 3 to state 1.
Consider passing from states 1 and 3 into state 2.

\[ \mathbf{m_2} = \begin{pmatrix} m_{12} \\ m_{32} \end{pmatrix} \]

Here we solve the system

\[
(I - N_2)\mathbf{m_2} = \mathbf{1}
\]

\[ \Rightarrow \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.3 & 0.1 \\ 0.05 & 0.55 \end{pmatrix} \right) \begin{pmatrix} m_{12} \\ m_{32} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ \mathbf{m_2} = \begin{pmatrix} 1.77419354839 \\ 2.41935483871 \end{pmatrix} \]

Here we see that it will take about 1.7 transitions to get from state 1 to state 2, and 2.4 transitions to get from state 3 to state 2.

Consider passing from states 1 and 2 into state 3.

\[ \mathbf{m_3} = \begin{pmatrix} m_{13} \\ m_{23} \end{pmatrix} \]

Here we solve the system

\[
(I - N_3)\mathbf{m_3} = \mathbf{1}
\]

\[ \Rightarrow \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.3 & 0.6 \\ 0.1 & 0.6 \end{pmatrix} \right) \begin{pmatrix} m_{13} \\ m_{23} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ \mathbf{m_3} = \begin{pmatrix} 4.54545454545 \\ 3.63636363636 \end{pmatrix} \]

The mean first passage time tells us that it will take approximately 4.54 transitions to move from state 1 to state 3, and 3.63 transitions to move into state 3 from state 2.

Note the mean first passage time from state \( i \) back to state \( i \) is given by

\[ m_{ii} = \frac{1}{\pi_i} \]

giving us a full set of mean first passage times. Thus, the mean first passage time to transition from a given state back to that particular state for our example problem is:

\[ m_{11} = 9.83333333333, \ m_{22} = 1.90322580645, \ and \ m_{33} = 2.68181818182. \]
6.1.6 Absorption Probabilities

There are many instances in which we would wish to determine the expected number of steps it will take a Markov chain to reach an absorbing state. Examples could be a garden’s soil year to year soil quality, with a poor soil garden being an absorbing state. You could model a person’s position within a corporation with retirement being an absorbing state. The gambler’s ruin problem discussed earlier also had two absorbing states.

Consider the following probability transition matrix:

\[
P = \begin{pmatrix}
0 & 0.2 & 0.3 & 0.4 & 0.1 \\
0 & 1.0 & 0.0 & 0.0 \\
0.5 & 0.0 & 0.0 & 0.0 & 0.2 \\
0.0 & 0.0 & 0.0 & 0.0 & 1.0
\end{pmatrix}
\]

Note that we may rearrange this system into the following transition matrix by reordering the states from

\[1, 2, 3, 4 \Longrightarrow 1, 3, 2, 4\]

This yields:

\[
P^* = \begin{pmatrix}
0.2 & 0.4 & 0.3 & 0.1 \\
0.5 & 0.0 & 0.3 & 0.2 \\
0.0 & 0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 1.0
\end{pmatrix}
\]

where the absorbing states of the system are all located in the lower right hand portion of the transition matrix.

Note that \( P^* \) has the following form:

\[
P^* = \begin{pmatrix}
N \\ A
\end{pmatrix} \begin{pmatrix}
I
\end{pmatrix}
\]

where

\[
N = \begin{pmatrix}
0.2 & 0.4 \\
0.5 & 0.0
\end{pmatrix}, \quad A = \begin{pmatrix}
0.3 & 0.1 \\
0.3 & 0.2
\end{pmatrix}, \quad \text{and} \quad I = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

When a transition matrix is arranged in the manner described above, and using \( \mathbf{1} \) to denote a column vector of ones it may be shown that:

- \( (I - N)^{-1} \) gives the expected time in state \( j \) starting in state \( i \).
- \( (I - N)^{-1} \mathbf{1} \) gives the expected time to absorption vector starting in state \( i \)
- \( (I - N)^{-1} A \) gives the probability of absorption in state \( j \) given starting state \( i \).

Using our sample matrix:

\[
(I - N)^{-1} = \begin{pmatrix}
0.6667 & 0.6667 \\
0.8333 & 1.3333
\end{pmatrix} \quad \text{and} \quad (I - N)^{-1} A = \begin{pmatrix}
0.7 & 0.3 \\
0.65 & 0.35
\end{pmatrix}
\]
- The average number of visits for something starting in original state 1 to state 1 is 5/3, and the number of visits for something starting in original state 1 to state 3 before being absorbed is 2/3.

- There is a 70% probability that something starting in original state 1 is going to be absorbed by original state 2.

- There is a 35% probability that something starting in original state 2 is absorbed by original state 4.

The matrix \((I - N)^{-1}\) is often called a Markov chain’s fundamental matrix. For proof of the above the interested reader is referred to Resnick’s text [5].

**Example**

The following example is from Taha’s Text: A product is processed on two sequential machines, I and II. Inspection takes place after a product unit is completed on either machine. There is a 5% chance that the unit will be junked before inspection. After inspection, there is a 3% chance the unit will be junked, and a 7% chance of being returned to the same machine for reworking. Else, a unit passing inspection on both machines is good.

- For a part starting at machine I, determine the average number of visits to each state.

- If a batch of 1000 units is started on machine I, determine the average number of completed good units.

- If the processing time at machine I and II is 20 and 30 minutes respectively, and the inspection times at machines I and II are 5 and 7 minutes what is the expected processing time for a part starting at machine I?

From the problem description we can see that the states in the process are:

- start I, inspect I, start II, inspect II, Junk, Good

Note that Junk and Good will be absorbing states in our Markov chain. Ordering the states in this manner the probability transition matrix is:

\[
P = \begin{pmatrix}
0.0 & 0.95 & 0.0 & 0.0 & 0.05 & 0.0 \\
0.07 & 0.0 & 0.9 & 0.0 & 0.03 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.95 & 0.05 & 0.0 \\
0.0 & 0.0 & 0.07 & 0.0 & 0.03 & 0.9 \\
0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0
\end{pmatrix}
\]

Separating out \(N\) and \(A\) are then:

\[
N = \begin{pmatrix}
0.0 & 0.95 & 0.0 & 0.0 \\
0.07 & 0.0 & 0.9 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.95 \\
0.0 & 0.0 & 0.07 & 0.0
\end{pmatrix}
\]

\[
A = \begin{pmatrix}
0.05 & 0.0 \\
0.03 & 0.0 \\
0.05 & 0.0 \\
0.03 & 0.9
\end{pmatrix}
\]
We can now compute the fundamental matrix:

\[
(I - N)^{-1} = \begin{pmatrix}
1.0712 & 1.0177 & 0.9812 & 0.9321 \\
0.075 & 1.0712 & 1.0328 & 0.9812 \\
0.0 & 0.0 & 1.0712 & 1.0177 \\
0.0 & 0.0 & 0.075 & 1.0712 \\
\end{pmatrix}
\]

and

\[
(I - N)^{-1}A = \begin{pmatrix}
0.1611 & 0.8389 \\
0.117 & 0.883 \\
0.0841 & 0.9159 \\
0.0359 & 0.9641 \\
\end{pmatrix}
\]

Form the given computations we can now make some observations:

- The probability of a unit that is starting at machine I being junked is 16.1%, and being good is approximately 83.9%.
- Note the probability of a unit being inspected after machine II has a 3.6% chance of being Junk, and a 96.4% chance of being good.

To answer the questions posed at the outset. Parts starting at machine I visit Machine I approximately 1.07 times, Inspection I approximately 1.02 times, Machine II approximately 0.98 times, and the second inspection approximately 0.93 times.

To answer the processing time for a good unit we have:

\[
1.0712(20) + 1.0177(5) + 0.9812(30) + 0.9321(7) = 62.4732000000000.
\]

So the processing time for a good unit that starts in state I is approximately 62.5 minutes.
Example:

When I borrow a book for the city library, I try to return it after one week. Depending on the length of the book and my free time, there is a 30% chance that I keep it for another week. If I have had a book for two weeks, there is a 10% chance that I’ll keep the book for an additional week. Under no conditions do I keep it for more than three weeks.

• Express the situation as a Markov chain. The transition matrix is:

\[
\begin{pmatrix}
0 & 0.3 & 0.0 & 0.7 \\
0 & 0.0 & 0.1 & 0.9 \\
0 & 0.0 & 0.0 & 1.0 \\
0 & 0.0 & 0.0 & 1.0 \\
\end{pmatrix}
\]

when we consider the rows in the order of:

Out 1, Out 2, Out 3, Returned

• Determine the average number of weeks before returning a book to the library.

\[
N = \begin{pmatrix}
0.0 & 0.3 & 0.0 \\
0.0 & 0.0 & 0.1 \\
0.0 & 0.0 & 0.0 \\
\end{pmatrix}
\]

\[
I - N = \begin{pmatrix}
1.0 & -0.3 & 0.0 \\
0.0 & 1.0 & -0.1 \\
0.0 & 0.0 & 1.0 \\
\end{pmatrix} \implies (I - NL)^{-1} = \begin{pmatrix}
1.0 & 0.3 & 0.03 \\
0.0 & 1.0 & 0.1 \\
0.0 & 0.0 & 1.0 \\
\end{pmatrix}
\]

To find the expected time to absorption we have:

\[
m_1 = (I - N)^{-1}1 = (1.33, 1.1, 1.0)
\]

To fine the probability of absorption in the given state we have:

\[
(I - N)^{-1}A = \begin{pmatrix}
1.0 & 0.3 & 0.03 \\
0.0 & 1.0 & 0.1 \\
0.0 & 0.0 & 1.0 \\
\end{pmatrix} \begin{pmatrix}
0.7 \\
0.9 \\
1.0 \\
\end{pmatrix} = \begin{pmatrix}
1.0 \\
1.0 \\
1.0 \\
\end{pmatrix}
\]